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I. INTRODUCTION

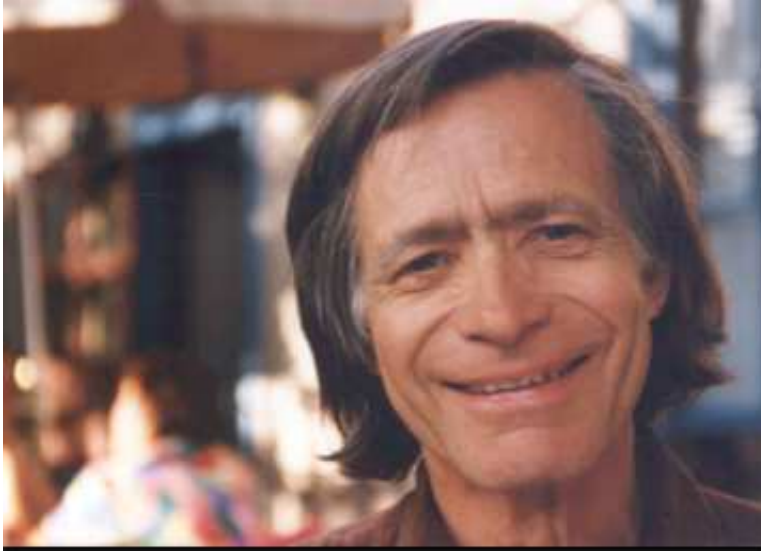


Figure 1 Robert H. Kraichnan. Photo by Judy Moore-Kraichnan

Robert Harry Kraichnan (1928–2008) was one of the leaders in the theory of turbulence for a span of about forty years (mid-fifties to mid-nineties). Among his many contributions, he is perhaps best known for his work on the inverse energy cascade (i.e. from small to large scales) for forced two-dimensional turbulence. This discovery was made in 1967 at a time when two-dimensional flow was becoming increasingly important for the study of large-scale phenomena in the Earth’s atmosphere and oceans. The impact of the discovery was amplified by the development of new experimental and numerical techniques that allowed full validation of the conjecture.

How did Kraichnan become interested in turbulence? His earliest scientific interest was in general relativity, which he began to study on his own at age 13. At age 18 he wrote at MIT a prescient undergraduate thesis, “Quantum Theory of the Linear Gravitational Field”; he received a PhD in physics from MIT in 1949 for his thesis, “Relativistic Scattering of Pseudoscalar Mesons by Nucleons,” supervised by Herman Feshbach. His interest in turbulence arose in 1950 while assisting Albert Einstein in search for highly nonlinear, particle-like solutions to unified field equations. He wrote to his long-time friend and collaborator J.R. Herring in the early nineties how this happened:

I realized I had no real idea of what I was doing and turned to Navier–Stokes as a nonlinear field problem where experiment could confront speculation. After initial surprise that turbulence did not succumb rapidly to field-theoretic attack, I have been trapped ever since. My overall research theme regarding

turbulence has been to understand what aspects of turbulence can and cannot be described by statistical mechanics; that is, what characteristics follow from invariances, symmetries, and simple statistical measures, and what, in contrast, can be known only from experiment or detailed solution of the equations of motion. The principal tool I have used is the construction of a variety of stochastic dynamical systems that incorporate certain invariances and other dynamical properties of actual turbulence but whose statistics are exactly soluble. The similarities and differences between these model solutions and real turbulence help illuminate what is essential in turbulence dynamics. My interests have centered on isotropic Navier–Stokes and magnetohydrodynamic turbulence in two and three dimensions.

When Kraichnan started working in turbulence, the community was trying to turn A.N. Kolmogorov’s 1941 (K41) ideas into a quantitative theory devoid of adjustable parameters. Kraichnan was the only one who truly succeeded in this endeavor, around the early sixties, by making among other things extensive use of his training in theoretical physics and field theory. By that time it had however become clear that K41 is not the final word, because it misses intermittency effects that produce anomalous scaling, that is, scaling laws whose exponents cannot be obtained by dimensional considerations. Kraichnan’s last major work proposed a fluid-dynamical framework for studying intermittency that quickly led the younger generation to identify the mathematical mechanism of intermittency and also to calculate by perturbation theory the anomalous scaling exponents for Kraichnan’s passive scalar model.¹

These were forty difficult years, particularly the first twenty when K41 seemed to collapse, no *ab initio* theory was emerging and direct numerical simulation had not yet reached Reynolds numbers high enough to supplement experimental data on intermittency. Thus a kind of crossing of the desert took place, with Kraichnan leading a small flock that found help mostly from geophysicists (at the Wood Hole Oceanographic Institute’s GFD Program and at the National Center for Atmospheric Research, Boulder).

This crossing of the desert occurred at a time when Kraichnan took himself “far from the madding crowd”. After having trod the traditional path of academia with positions at Columbia University and Courant Institute, he decided in the very early 60’s to become a self-employed turbulence consultant, solely funded by research grants. He moved from New York City to secluded mountains in New Hampshire, where he lived and worked for almost two decades. During those years he warmly welcomed visitors from all around the world and regularly participated in scientific meetings, workshops and schools. Eventually, he moved to New Mexico, close to the Los Alamos National Laboratory.

We shall take the reader on a tour of key contributions to turbulence by Kraichnan, organized in three main sections, roughly by chronological order: closure, realizability, the issue of Galilean invariance and MHD turbulence (Sec. II); equilibrium statistical mechanics and two-dimensional turbulence (Sec. III); intermittency and the Kraichnan passive-scalar model (Sec. IV). In the final section (Sec. V), we first present two contributions of Kraichnan which do not fit naturally into the three main scientific sections: his very first published paper, on scattering of sound by turbulence (Sec. V.A), and his prediction of the behavior of convection at extremely high Rayleigh numbers (Sec. V.B). We then turn to Kraichnan’s impact on computational turbulence and present concluding remarks.

The emphasis will be on understanding the flow of ideas and how they relate to the meandering history of the subject. We shall mostly avoid technical material. More details may be found in Kraichnan’s numerous papers (over one hundred), published in journals such as *Phys. Rev.*, *Phys. Fluids*, *J. Fluid Mech.* and *J. Atmosph. Sci.*, and in various reviews and textbooks by other authors,² but also in proceedings papers by Kraichnan, which are often more elementary and reader-friendly than the journal papers.³

A word of warning: in the sequel, standard notions used in turbulence theory, such as the

¹ On K41, cf. Kolmogorov, 1941; Frisch, 1995.

² Leslie, 1973; Monin & Yaglom, 1975: § 19.6; Orszag, 1977; Rose & Sulem, 1978.

³ Cf. Kraichnan, 1958c, 1972, 1975a.

Navier–Stokes and the magnetohydrodynamics (MHD) equations, statistical homogeneity and isotropy, energy spectra, etc will be taken for granted.⁴

II. CLOSURES: REALIZABILITY, GALILEAN INVARIANCE AND THE RANDOM COUPLING MODELS; MHD TURBULENCE

After World War II the K41 theory became universally known and was seen to be mostly consistent with experimental data of that time. Attempts were then made to build a quantitative theory of homogeneous and isotropic turbulence, compatible with K41 but free of undetermined constants, such as the (Kolmogorov) constant appearing in the energy spectrum and able to predict the functional form of the spectrum in the dissipation range. Particularly noteworthy was S. Chandrasekhar’s attempt, based on the quasi-normal approximation (QNA). Introduced by M.D. Millionschikov, the QNA assumes that second- and fourth-order moments are related as in a normal (Gaussian) distribution. This is an instance of what is called *closure*: replacing the infinite hierarchy of moment equations derivable from the Navier–Stokes equations by a finite number of equations for suitable statistical quantities such as two-point correlations. Kraichnan was the first to point out that the simplest closures that come to mind, such as the QNA can have *realizability* problems: basic probabilistic inequalities may be violated. A few years later, using the form of the QNA obtained by T. Tatsumi, Y. Ogura showed indeed that it can lead to an energy spectrum which is negative for certain wavenumbers.⁵

A. The Direct Interaction Approximation and the Galilean problem

Kraichnan thus set out to obtain better closures, simultaneously free of adjustable parameters and realizable. He was the first to apply field-theoretic methods to non-equilibrium statistical fluid dynamics (Navier–Stokes and MHD): he observed that formal expansions in powers of the nonlinearity of the solutions of such equations, followed by averaging over random initial conditions and/or driving forces, generates terms which can be represented by Feynman diagrams. Eventually, P.C. Martin, E.D. Siggia and H.A. Rose would establish a detailed formal connection between statistical dynamics of classical fields, such as turbulence dynamics, and a certain quantum field theory (so-called MSR field theory).⁶

In tackling the closure problem from this point of view, Kraichnan was led to introduce a new object into turbulence theory, namely the (averaged infinitesimal) response function, that is, the linear operator giving the small change in the velocity field at time t , due to a small change in the driving force at times $t' < t$. In 1958, using a mixture of field-theoretic considerations and heuristic but plausible simplifying assumptions, Kraichnan proposed the Direct Interaction Approximation (DIA).⁷ When applied to homogeneous, isotropic and parity-invariant turbulence, the DIA takes the form of two coupled integro-differential equations for the two-point two-time velocity correlation function and the response function. These equations are written in Sec. II.B in abstract and concise notation, valid without assuming any particular symmetry. For homogeneous isotropic turbulence the DIA equations, which resemble in more complicated form the EDQNM equation (3) written below, can be found in Kraichnan’s papers and in the references mentioned in the Introduction. Such equations are easily extended to anisotropic and/or helical (non-parity-invariant) turbulence.

As Kraichnan observed himself, the DIA is not consistent with K41. In particular the inertial-range energy spectrum predicted by DIA is (to leading order)

$$E(k) = C'(\varepsilon v_0)^{1/2} k^{-3/2}, \quad (1)$$

⁴ Cf., e.g., Monin & Yaglom, 1975; Frisch, 1995.

⁵ Millionschikov, 1941; Chandrasekhar, 1955; cf. also Spiegel, 2010; Kraichnan, 1957b; Tatsumi, 1957; Ogura, 1963.

⁶ Kraichnan, 1958c; Martin, Siggia & Rose, 1973.

⁷ Kraichnan, 1958a, 1958c, 1959a.

instead of

$$E(k) = C\varepsilon^{2/3}k^{-5/3}, \quad (2)$$

where ε is the kinetic energy dissipation per mass, v_0 is the r.m.s. turbulent velocity, and C (the Kolmogorov constant) and C' are dimensionless constants. The discrepancy has to do with the way the larger eddies from the energy-containing range of wavenumbers interact with smaller eddies from the inertial range. Kraichnan expressed his doubts that the former merely

convect local regions of the fluid bodily without significantly distorting their small-scale (high- k) internal dynamics.

This assumption would not permit the presence of v_0 in the inertial-range expression of the energy spectrum, except through an intermittency effect of a kind which was not seriously considered until the work by the Russian School in the early sixties. Nevertheless, Kraichnan observed the following:

As a counter-consideration to Kolmogorov's original argument in the x -space representation, it may be noted that the fine-scale structure of high Reynolds-number turbulence consists typically not of compact blobs but of a complicated tangle of extended vortex filaments and sheets.⁸

For a while Kraichnan believed that the DIA may be asymptotically exact in the limit $L \rightarrow \infty$ for turbulence endowed with L -periodicity in all the spatial coordinates. However, after a few months, Kraichnan found that the DIA equations are actually the exact consequences of a certain random coupling model obtained from the hydrodynamical equations (Navier–Stokes or MHD) by a suitable modification of the nonlinear interactions (see Sec. II.B). The presence of this model automatically guarantees the realizability of the DIA. The random coupling model has many properties in common with the original hydrodynamical equations, such as symmetries, energy conservation, etc. But Kraichnan noted that, when various correlation functions are expanded in terms of Feynman diagrams, the random coupling model retains only a subclass of all diagrams; in particular it misses the so-called vertex corrections which contribute (in field-theoretic language) to the renormalization of the nonlinear interaction.⁹

Six years later he discovered a more serious defect: the random coupling model and the DIA fail invariance under *random Galilean transformations*.¹⁰ Ordinary Galilean invariance — for the Navier–Stokes equations — is the observation that if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ are the velocity and pressure fields which solve the Navier–Stokes equations in the absence of boundaries or with periodic boundary conditions, then $(\mathbf{u}(\mathbf{x} - \mathbf{V}t, t) + \mathbf{V}, p(\mathbf{x} - \mathbf{V}t, t))$ are also solutions for an arbitrary choice of the velocity \mathbf{V} . Random Galilean invariance has the velocity \mathbf{V} chosen randomly with an isotropic distribution and independent of the turbulent velocity-pressure fields. Isotropy ensures that the mean velocity remains zero. In the DIA, when a random Galilean transformation is performed, the mean square velocity v_0^2 increases and thus the spectrum given by (1) changes. On the one hand this is clearly inconsistent with the Galilean invariance of the Navier–Stokes equations. On the other hand, an influence of the energy-carrying eddies on the inertial range different from what K41 predicts (e.g. intermittency effects) cannot be ruled out.

But first, Kraichnan realized that the DIA had to be modified to restore random Galilean invariance and, if possible, without losing its other nice features. To explain how this can be done, we have to become slightly more technical. Kraichnan derives the following equation

⁸ Kraichnan, 1958a: §§ 9-10 and footnote 35.

⁹ Kraichnan, 1958b, 1958c.

¹⁰ Kraichnan, 1964b; Note that the random coupling model can be modified to preserve a Galilei symmetry group of dimension 3, for a model with N 3-dimensional velocity fields. See Kraichnan, 1964a: Appendix. However, this is insufficient to guarantee random Galilean invariance for the DIA equations.

from the DIA for the evolution of the energy spectrum when the turbulence is homogeneous, isotropic and parity-invariant:

$$(\partial_t + 2\nu k^2) E(k, t) = \int \int_{\Delta_k} dp dq \theta_{kpq} \times \\ b(k, p, q) \frac{k}{pq} E(q, t) [k^2 E(p, t) - p^2 E(k, t)] + F(k), \quad (3)$$

$$\theta_{kpq} \equiv \frac{(\pi/2)^{1/2}}{v_0(k^2 + p^2 + q^2)^{1/2}}, \quad b(k, p, q) = \frac{p}{k}(xy + z^3). \quad (4)$$

Here $E(k, t)$ is the energy spectrum, $F(k)$ the energy input from random driving forces, Δ_k defines the set of $p \geq 0$ and $q \geq 0$ such that k, p, q can form a triangle, x, y, z are the cosines of the angles opposite to sides k, p and q of this triangle. The factor θ_{kpq} can be interpreted as a triad relaxation time. The particular form given in (4) is obtained from the DIA by an asymptotic expansion valid only when the wavenumber k is in the inertial range. This form implies that relaxation has a time scale comparable to the convection time $1/(kv_0)$ of an eddy of size $1/k$ at the r.m.s. velocity v_0 . As Kraichnan notes, if v_0 is replaced “by, say, $[kE(k)]^{1/2}$, which may be considered the r.m.s. velocity associated with wavenumbers the order of k only”, then K41 is recovered. Actually this choice, which also restores random Galilean invariance and is easily shown to be realizable, became later known as the Eddy Damped Quasi-Normal Markovian (EDQNM) approximation. Note that the EDQNM requires the use of an adjustable dimensionless constant in front, say, of $[kE(k)]^{1/2}$, which can then be tuned to give a Kolmogorov constant matching experiments and/or simulations. Kraichnan clearly preferred having a systematic theory free of such constants and then seeing how well it can reproduce experimental data, not just constants, but also the complete functional forms of spectra. He eventually developed a version of the EDQNM, called the Test Field Model in which the triad relaxation time is determined in a systematic way through the introduction of a passively advected (compressible) test velocity field and a procedure for eliminating unwanted sweeping effects.¹¹

To overcome the difficulty with random Galilean invariance in a more systematic way, however, he first developed the Lagrangian History Direct Interaction Approximation (LHDIA) and an abridged version thereof (ALHDIA) in which the correlation functions have fewer time arguments. LHDIA and ALHDIA make use of a generalized velocity $\mathbf{u}(\mathbf{x}, t|t')$ which has both Eulerian and Lagrangian characteristics. It is defined as the velocity measured at time t' in that fluid element which passes through \mathbf{x} at time t , and it satisfies an extended form of the Navier–Stokes equation in the two time variables. When the DIA closure is written for this extended system there appear integrals over the past history of the flow, which can be altered when working with the generalized field in such a way as to recover random Galilean invariance. Detailed numerical studies of the ALHDIA gave a full functional form of the spectra in very good agreement with experimental data. In particular the Kolmogorov constant was $C = 1.77$ while the best current value is 1.58, when ignoring intermittency corrections. Other detailed predictions were made for various Lagrangian quantities involving correlations of velocity increments, pressure-gradients, particle accelerations and also for 2-particle dispersion. Many of these predictions are still to be subjected to experimental test.¹²

In the course of studying the DIA and all its “children”, Kraichnan had to develop quite sophisticated analytic and numerical tools. For example he showed that the spectrum falls off exponentially in the far dissipation range and obtained the algebraic prefactor in front of the exponential. Such tools, are transposable almost immediately to the study of other

¹¹ Kraichnan, 1958c; on EDQNM, cf. Orszag, 1966, 1977 and see also Rose & Sulem, 1978; Lesieur, 2008; on the Test Field Model, cf. Kraichnan, 1971a, 1971b.

¹² Kraichnan, 1964b, 1965a, 1966a; on experimental data, cf. Grant, Stewart & Moilliet, 1962; on the Kolmogorov constant, cf. Donzis & Sreenivasan, 2010. Ott & Mann, 2000, corrected an error in Kraichnan’s LHDIA theory of turbulent two-particle dispersion (Kraichnan, 1966b) and found from experiment that its prediction for the Richardson-Obukhov constant was too large by a factor of 10.

closures, for example of the kinetic theory of resonant nonlinear wave interactions.¹³ They are also part of the Kraichnan legacy.

Many of Kraichnan's tools from statistical mechanics and field theory were unfamiliar to fluids scientists trained in the traditions of theoretical mechanics and applied mathematics, and some objects central to the DIA —particularly the mean response function—did not appear in previous exact moment hierarchies. The DIA was thus received with some scepticism by the scientific community, in particular in Cambridge (UK), but Kraichnan was given the opportunity to present his theory in the recently introduced *Journal of Fluid Mechanics*. Kraichnan's theory was discussed by Proudman at the famous Marseille 1961 conference. With hindsight, we see that he failed to notice the realizability of the DIA and lumped it together with the quasi-normal approximation (QNA); actually, it would take two more years until Ogura discovered the aforementioned problems with QNA. Much more interesting is that, at the end of his paper, Proudman (correctly) expressed doubts that DIA could cope with what was later termed dissipation-range intermittency, which would be first explained by Kraichnan six years later (cf. Sec. IV.A).¹⁴ Eventually, Kraichnan's efforts to develop an analytical theory, compatible with K41 and able to give detailed quantitative predictions, was remarkably successful and influential. Of course around the same time, at the Marseille conference, it became increasingly clear that K41 was not the final word for fully developed turbulence. Kraichnan's contributions to post-K41 theories will be described in Sec. IV.

B. The random coupling models and the $1/N$ expansion

Starting from the Navier–Stokes (NS) equations Kraichnan derived the DIA, using heuristic assumptions, that even if plausible, turned out to be partially false. Quite rapidly after this, Kraichnan showed that the DIA equations are actually the *exact* consequence in a certain limit of a stochastic model, called the random coupling model (RCM), which is obtained by a suitable modification of the Navier–Stokes equations. This is important not only because it ensures the realizability of the DIA but because it makes contact with—and largely anticipates—techniques still being developed in field theory and statistical mechanics. We shall try here to give a not-too-technical presentation of the RCM.¹⁵

Random coupling models can be written for a large class of quadratically nonlinear partial differential equations, which encompass the Burgers equation, the Navier–Stokes equations and the MHD equations. All these can be written in the following general form, where it is assumed that the solution is zero for $t < 0$:

$$\partial_t u(t) + B(u(t), u(t)) = Lu(t) + f(t) + u(0)\delta(t). \quad (5)$$

Here $u(t)$ (scalar or vector) collects all the dependent variables, $B(.,.)$ is a quadratic form (comprising the inertial and pressure terms for the case of Navier–Stokes), L is a time-independent linear operator (e.g., the viscous dissipation operator) and $f(t)$ is a prescribed zero-mean random force, taken Gaussian for convenience; the prescribed zero-mean initial condition $u(0)$ in front of the Dirac distribution $\delta(t)$ is also taken random Gaussian. Obviously, (5) can be rewritten as an equivalent integral equation

$$u(t) + \int_0^t dt' e^{L(t-t')} B(u(t'), u(t')) = \int_0^t dt' e^{L(t-t')} f(t') + u(0). \quad (6)$$

In somewhat more abstract form this may be written as

$$u + \mathcal{B}(u, u) = \mathcal{F}. \quad (7)$$

Let us now describe the random coupling model in the form used by Herring and Kraichnan. Imagine that N independent replicas of (7) are written with N independent fields,

¹³ Zakharov, L'vov & Falkovich, 1992; Nazarenko, 2010.

¹⁴ Kraichnan, 1959a; Proudman, 1961; Ogura, 1963.

¹⁵ Kraichnan, 1958c, 1961.

labeled u_α ($\alpha = 1, \dots, N$) and that these are then coupled as follows:

$$u_\alpha + \frac{1}{N} \sum_{\beta, \gamma} \phi_{\alpha\beta\gamma} \mathcal{B}(u_\beta, u_\gamma) = \mathcal{F}_\alpha. \quad \alpha = 1, \dots, N. \quad (8)$$

Here, the \mathcal{F}_α are N independent identically distributed (iid) replicas of the force \mathcal{F} in (7); the “random coupling coefficients” $\phi_{\alpha\beta\gamma}$ are a set of Gaussian random variables of zero mean and unit variance. These N^3 coefficients are taken all independent, except for the requirement that $\phi_{\alpha\beta\gamma}$ be invariant under all permutations of (α, β, γ) , a condition crucial for conservation of energy and other quadratic invariants. When the number of replicas tends to infinity, closed equations emerge for the two following objects: the correlation function $U \equiv \langle u_\alpha \otimes u_\alpha \rangle$ and the infinitesimal response function $G \equiv \langle \delta u_\alpha / \delta \mathcal{F}_\alpha \rangle$. Note that off-diagonal terms with $\alpha \neq \beta$ tend to zero as $N \rightarrow \infty$. The angular brackets denote ensemble averages and \otimes indicates a tensor product: if we were dealing with a velocity field $u_i(\mathbf{x}, t)$, then $u \otimes u$ would stand for $u_i(\mathbf{x}, t) u_j(\mathbf{x}', t')$. It is also convenient to introduce auxiliary independent zero-mean Gaussian random fields \hat{v} and v having the same correlation function U as any of the u_α ’s. With such abstract notation the DIA equations take the compact form:

$$U = \langle (G\mathcal{F}) \otimes (G\mathcal{F}) \rangle + 2G \langle \mathcal{B}(\hat{v}, v) \otimes G\mathcal{B}(\hat{v}, v) \rangle \quad (9)$$

$$G - 4 \langle \mathcal{B}(\hat{v}, G\mathcal{B}(\hat{v}, G)) \rangle = I, \quad (10)$$

where I is the identity operator.¹⁶

The random coupling model can also be modified to give equations other than DIA. For example the phases $\phi_{\alpha\beta\gamma}$ can be made functions of the time with a white noise dependence: $\langle \phi_{\alpha\beta\gamma}(t) \phi_{\alpha\beta\gamma}(t') \rangle = \tau_0 \delta(t - t')$; this is the so-called Markovian Random Coupling Model (MRCM) which leads to an equation similar to (3) but with $\theta_{kpq} \equiv \tau_0/4$. The EDQNM equation can also be obtained in similar manner by taking the random phase dependent on triads of nonlinearly interacting wavevectors and delta-correlated in time with a coefficient proportional to the triad relaxation time θ_{kpq} chosen in agreement with K41, as explained in Sec. II.A. Unfortunately, so far no random coupling model has been found for the Lagrangian variants of DIA.¹⁷

Kraichnan’s work on the RCMs has many other interesting connections in mathematics and physics. For example, the simplest dynamics considered in Kraichnan’s 1961 paper is the classical harmonic oscillator with a random frequency. The N -replica model in this case reduces to the study of a class of random $N \times N$ matrices (Toeplitz matrices with coefficients whose phases are randomly altered, consistent with Hermiticity). Following a suggestion of Kraichnan, it was shown by U. Frisch and R. Bourret that the DIA equation for the harmonic oscillator can also be obtained from an RCM employing random Wigner matrices selected out of the Gaussian Unitary Ensemble. This approach works for any linear dynamics with a random linear operator, such as the problems of turbulent advection of a passive scalar or the Schrödinger equation with random potential.

Kraichnan’s discovery of the RCM in 1958 anticipated, in fact, decades-later developments in quantum field theory employing large- N models with quenched random parameters and random matrices. $SU(N)$ gauge field theories for $N \rightarrow \infty$ were shown by G. ’t Hooft to resum a subset of all Feynman diagrams, the planar diagrams, and Yu.M. Makeenko and A.A. Migdal showed that the exact Schwinger–Dyson equations for Wilson loops reduce in that limit to a closed set of self-consistent equations like DIA. The most direct connection with Kraichnan’s work is for random matrix models of large- N gauge theory in 0D and quantum gravity in 2D, where the self-consistent “loop equation” is identical in form to the DIA equation for the harmonic oscillator with a non-Gaussian random frequency.¹⁸

¹⁶ On the random coupling model, cf. Kraichnan, 1958c, 1961, Herring & Kraichnan, 1972; on the compact form of the DIA, cf. Lesieur, Frisch & Brissaud, 1971.

¹⁷ On variants of the random coupling model, cf. Kraichnan, 1958c, 1961; on the MRCM, cf. Frisch, Lesieur & Brissaud, 1974.

¹⁸ Kraichnan, 1961: § 9.

Kraichnan's RCM was not a rote application of standard field-theoretic techniques of the 1950's, but employed advanced ideas that were regarded as "cutting edge" decades later.¹⁹

The Random Coupling Models may have more than just historical interest, however. One of the successes of theoretical physics in the 1970's was the development of $1/N$ expansion methods to provide analytical tools to tackle nonperturbative problems with no other small parameter, such as anomalous scaling in critical phenomena and quark confinement in quantum chromodynamics. As we shall discuss in more detail in Sec. IV, turbulence anomalous scaling due to inertial-range intermittency is exactly this sort of problem. Moreover, recent numerical studies of large- N Random Coupling Models for some simple dynamical systems model of turbulence, so-called "shell models", have found that inertial-range anomalous dimensions vanish proportional to $1/N$. Kraichnan's pioneering work on the RCM together with modern $1/N$ expansion methods might provide a powerful tool to address the difficult problem of inertial-range turbulence scaling.²⁰

C. MHD turbulence

Magnetohydrodynamics (MHD) was born with Alfvén's prediction of a new type of waves (now called Alfvén waves) in a conducting fluid in the presence of a uniform background magnetic field. As pointed out by von Neumann in his report on turbulence at the conference "Problems of Motion of Gaseous Masses of Cosmic Dimensions," MHD was considered key in understanding the origin of various cosmic magnetic fields — beginning with that of the Earth — and the mechanism for accelerating cosmic rays. Immediately the question arose if MHD turbulence at high kinetic and magnetic Reynolds numbers can be described using K41. The question was difficult because (i) there were no experimental data on MHD turbulence in the late forties and (ii) dimensional analysis was of limited use, due to the presence of two fields, the velocity field \mathbf{u} and the magnetic field \mathbf{b} .²¹ Important early papers on the statistical theory of turbulence were already dealing with MHD turbulence, even in their titles. This included Batchelor's paper on the analogy between vorticity and magnetic fields, Lee's 1952 paper (cf. Sec. III) and Kraichnan's very first paper on the DIA, where he also handled the MHD case, but without explicitly stating that it leads to a $k^{-3/2}$ inertial-range spectrum, as in the purely hydrodynamic case.²²

One year after discovering that the DIA fails random Galilean invariance, Kraichnan returned to MHD and found that the $k^{-3/2}$ law may actually apply to the three-dimensional MHD case when there is a significant random large-scale magnetic field of r.m.s. strength b_0 . In a short research note he stated:

In the present hydromagnetic case, it still may be argued plausibly that the action of the energy range on the inertial range is equivalent to that of spatially uniform fields. But, in contrast to a uniform velocity field, a uniform magnetic field has a profound effect on energy transfer.

Indeed, when the equations of MHD are rewritten in terms of the Elsässer variables $\mathbf{z}^\pm \equiv \mathbf{v} \pm \mathbf{b}$, and a uniform background field of strength b_0 is present, it is found (i) that in linear theory \mathbf{z}^+ and \mathbf{z}^- propagate as Alfvén waves in opposite directions with speed b_0 along the background field, (ii) the only nonlinear interactions are between \mathbf{z}^+ and \mathbf{z}^- (no self-interactions). It follows that a \mathbf{z}^+ -eddy and a \mathbf{z}^- -eddy of wavenumbers $\sim k$ will be able to significantly interact only during a coherence time $\sim 1/(kb_0)$. Thus the flux ε of total (kinetic plus magnetic) energy through the wavenumber k will be proportional to $1/b_0$, where b_0 is the r.m.s. magnetic field fluctuation (stemming from the energy range). Therefore ε

¹⁹ On large- N gauge theory and quantum gravity, G. 't Hooft, 1974; Makeenko & Migdal, 1979; Migdal, 1983; Di Francesco, Ginsparg & Zinn-Justin, 1995.

²⁰ On DIA and Wigner matrices, cf. Frisch & Bourret, 1970. On $1/N$ expansion in critical phenomena, cf. Abe, 1973. On large- N shell models, cf. Pierotti, 1997; Pierotti, L'vov, Pomyalov & Procaccia, 2000.

²¹ The magnetic field can be given the same dimension as the velocity field after division by $\sqrt{4\pi\mu\rho}$ where μ is the magnetic permeability and ρ the density; these are the units used here.

²² Alfvén, 1942; von Neumann, 1949; Batchelor, 1950; Lee, 1952; Kraichnan, 1958a.

and b_0 must appear in the combination εb_0 . It then follows by dimensional analysis that both the kinetic and magnetic energy spectra are of the form

$$E(k) \sim (\varepsilon b_0)^{1/2} k^{-3/2}. \quad (11)$$

This is the same functional form as the DIA spectrum (1), and for a good reason: the large-scale magnetic field plays now the same role as the large-scale velocity field in the DIA. Of course, in the DIA this is a spurious role, whereas in MHD Alfvén waves make this effect real. It must be pointed out that Iroshnikov proposed the same MHD spectrum (11) slightly before Kraichnan, using a semi-phenomenological theory of interaction of three Alfvén waves and a diffusion approximation in k -space for the energy spectrum derived by arguments resembling those Leith used for two-dimensional turbulence. Iroshnikov’s work did not penetrate much into the West until approximately 1990. Nowadays the energy spectrum (11) is called the Iroshnikov–Kraichnan spectrum.²³

Finally we mention that the arguments of Iroshnikov and Kraichnan can be questioned because local anisotropies are ignored: in a small region in which the (large-scale) magnetic field is \mathbf{B} , those small-scale eddies having wavevectors perpendicular to \mathbf{B} , or nearly so, are hardly affected by Alfvén waves. Hence some kind of quasi-two-dimensional MHD turbulence can emerge. A more systematic theory of weakly interacting Alfvén waves predicts a k^{-2} spectrum for MHD turbulence. The problem of determining the spectrum of strong MHD turbulence—even at the level of a Kolmogorov mean-field theory, ignoring intermittency—is still quite open at this time.²⁴

III. STATISTICAL MECHANICS AND TWO-DIMENSIONAL TURBULENCE

The essentially statistical character of turbulent flow has been clearly understood long before even the 1941 work of Kolmogorov, by such scientists as Lord Kelvin, G.I. Taylor, N. Wiener, T. von Kármán, and J.M. Burgers. To this day, probably the most successful statistical theory of any dynamical problem is the equilibrium statistical mechanics of Gibbs, Boltzmann and Einstein for classical and quantum Hamiltonian systems. It was natural then that the pioneers in turbulence theory would seek some guidance and inspiration from Gibbsian statistical mechanics. Notable is the attempt by Burgers to apply the maximum entropy principle to turbulent flows in a series of papers in 1929–1933. Unfortunately, it was pointed out by Taylor in 1935 that mean energy dissipation (and thus entropy production) does not vanish in the limit of high Reynolds numbers for typical turbulent flows. Turbulence is thus a fundamentally non-equilibrium problem to which the Boltzmann–Gibbs formalism is not *directly* applicable. Nevertheless, a judicious application of equilibrium statistical theory to hydrodynamics can still yield crucial hints and ideas on the behavior of turbulent flows and Kraichnan was one of the most masterful practitioners of this art. He used Gibbsian statistical predictions in his construction of closure theories and, most subtly, to help to divine the behavior of strongly disequilibrium turbulent cascades. This approach played an especially important role in Kraichnan’s development of his dual cascade picture of two-dimensional turbulence. For this reason, we shall treat these two subjects together.

A. Equilibrium Statistical Hydrodynamics

Prior to Kraichnan’s work, one of the most significant applications of Gibbsian statistical mechanics to hydrodynamics was in fact to two-dimensional fluids, by L. Onsager in a 1949 paper titled “Statistical Hydrodynamics”. Onsager noted that “two-dimensional convection, which merely redistributes vorticity” leads to energy conservation at infinite

²³ Kraichnan, 1965b; Iroshnikov, 1963; Leith 1967.

²⁴ On anisotropies, cf. Kit & Tsinober, 1971; Rüdiger, 1974; Shebalin, Matthaeus, & Montgomery; 1983. On ideas of using resonant wave interactions, cf. Ng & Bhattacharjee, 1996; Goldreich & Sridhar, 1997. On the systematic theory of resonant wave interactions, cf. Galtier *et al.*, 2000; Nazarenko, Newell & Galtier, 2001.

Reynolds number. Thus, equilibrium statistical mechanics may be legitimately applied to 2D Euler hydrodynamics, if ergodicity is assumed (as usual). Onsager studied in detail the microcanonical distribution of a “gas” of N Kirchhoff point-vortices in an incompressible, frictionless, 2D fluid. His most striking conclusion was that, for sufficiently high kinetic energies of the point-vortices, thermal equilibrium states of *negative* absolute temperature occur. The point-vortices condense into a single large-scale coherent vortex, which Onsager suggested could explain the “ubiquitous” appearance of such structures in quasi-2D flows.²⁵

A conservative system appears in any space dimension if the Euler equations or the MHD equations for ideal unforced fluid flow are Galerkin truncated, as first noted by Burgers and, later independently, by T.D. Lee and E. Hopf. For the case of space-periodic flow and in the abstract notation of Sec. II.B this amounts to replacing the nonlinear term in (5) by $P_G B(u(t), u(t))$, where P_G is the Galerkin projection. This operator is defined in terms of the spatial Fourier representation of $u(t)$ by setting to zero all Fourier components of wavenumbers $k > K_G$, the truncation wavenumber. The above authors showed then that, in suitable variables (based on the real and imaginary parts of the the Fourier amplitudes) the dynamics can be written as a set of ordinary real differential equations

$$\dot{q}_\alpha = F(q_1, \dots, q_N), \quad \alpha = 1, \dots, N, \quad (12)$$

with a Liouville theorem $\sum_\alpha \partial \dot{q}_\alpha / \partial q_\alpha = 0$, which expresses the conservation of volumes in the N -dimensional phase space. The hypothesis of ergodicity implies equipartition of energy among all degrees of freedom. In three dimensions, this led Lee and Hopf to a k^2 energy spectrum, with no divergence of the total energy because of truncation. Both recognized that, with viscosity added and absent truncation, things will be very different and that Kolmogorov’s $-5/3$ spectrum is expected. In the MHD case, Lee obtained the same spectrum for the magnetic field and an equipartition between kinetic and magnetic energy for every Fourier harmonic. He conjectured this extends to non-equilibrium MHD turbulence, inferring that Kolmogorov’s $-5/3$ spectrum should also hold in the MHD case.²⁶

This was roughly the situation when Kraichnan entered the field of turbulence, and statistical mechanical ideas dominated much of his early thinking, too.²⁷ His first work on the application of equilibrium statistical mechanics to turbulence was in a hardly cited 1955 paper on compressible turbulence in which he rediscovered, in a slightly different context, the 1952 results of Lee. In particular Kraichnan found the Liouville theorem and equipartition solutions (here, the equipartition is between kinetic and potential acoustic energies). Kraichnan’s preoccupation with statistical mechanics continued in his first DIA paper. He discussed there the Gibbs ensembles for the Galerkin-truncated equations but noted that “This artificial equilibrium case does *not* describe turbulence at infinite Reynolds number,” when a forcing term and a viscous damping term are added.

There is, however, one very original and definitive result on equilibrium statistical mechanics obtained in Kraichnan’s first paper on DIA. He established there a Fluctuation-Dissipation Theorem (FDT; called by him, correctly, “Fluctuation-Relaxation”) for conservative nonlinear dynamics in thermal equilibrium, generalizing previous results of H.B. Callen and co-workers. Kraichnan’s derivation in an Appendix to the 1958 paper was very ingenious. He showed that the FDT arises from the stability of the Gibbs measure for two replicas of the original dynamics under couplings that preserve the Liouville theorem and energy conservation. Recognizing the more general interest in this result, Kraichnan one year later published this argument separately for any dynamics that conserves both phase volume and an arbitrary energy function E . In the special case of the truncated Euler system considered in the 1958 DIA paper, Kraichnan’s FDT reduces to the statement that the 2-time correlation function U and the mean response function G are proportional, with the absolute temperature supplying the constant of proportionality. In this case, the DIA equations reduce to a single equation for G , which Kraichnan discussed in a separate section

²⁵ Onsager, 1949.

²⁶ Hopf, 1952; Lee, 1952.

²⁷ Kraichnan had discussions about statistical mechanics and the foundations of thermodynamics in the late fifties with B. Mandelbrot (Mandelbrot, private communication, 2010).

on “Nondissipative Equilibrium” in his paper. This is the same as the “mode-coupling theory” applied to equilibrium critical dynamics twelve years later by K. Kawasaki (who was aware of and influenced by Kraichnan’s earlier work on DIA).²⁸

Kraichnan returned to the statistical mechanics of Galerkin-truncated ideal flow in a 1975 article “Remarks on Turbulence Theory.” Although ostensibly a review article, this work—characteristically for Kraichnan—contained many original ideas and results not published elsewhere. Kraichnan pointed out that the “absolute equilibrium” for the inviscid truncated system in 3D had an interesting domain of physical applicability, describing the small thermal fluctuations of velocity in a fluid at rest. As an application of the equilibrium DIA equations, he showed that

...the truncated Euler system in thermal equilibrium exhibits a dynamical damping of low-wavenumber disturbances just like the viscous damping of the Navier–Stokes system at zero temperature. If k_{\max} is taken as some kind of intermolecular spacing scale or mean free path, then the truncated Euler system constitutes a nontrivial model of a molecular liquid, with the equilibrium excitation corresponding to normal molecular thermal energy.

In the same paper he also suggested that the truncated Euler system could support an energy cascade just as in the Navier–Stokes system, for “a statistical ensemble whose initial distribution is multivariate-normal, with all energy concentrated in wavenumbers the order of k_0 .” In 1989 he and S. Chen went much further:

“the truncated Euler system can imitate NS fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wave-number modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared with the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity.”

Actually, in 2005 very-high-resolution spectral simulations of the 3D Galerkin-truncated Euler equations showed that, when initial conditions are used that have mostly low-wavenumber modes, the truncated inviscid system tends at very long times to thermal equilibrium with energy-equipartition, but it has long-lasting transients which are basically the same as for viscous high Reynolds-number flow, including a K41-type inertial range. In other words, the high-wavenumber thermalized modes act as an artificial molecular micro-world.²⁹

B. Two-Dimensional Turbulent Cascades

All this is to set the stage for a major contribution made by Kraichnan in 1967, his theory of inverse cascade for two-dimensional (2D) turbulence. Equilibrium statistical mechanical reasoning is one of the many strands of thought that Kraichnan wove together into a compelling argument for the existence of Kolmogorov-type cascades in 2D incompressible fluids. His remarkable paper in *Physics of Fluids* invoked also an analysis of triadic interactions under the assumption of scale-similarity, exact results on the instantaneous evolution of Gaussian-distributed initial conditions, various plausible statistical and physical arguments, and even comparison with Kraichnan’s simultaneous work on dynamics of quantum Bose condensation. As noted already in the Introduction, Kraichnan’s findings have considerable interest for understanding large-scale geophysical and planetary flows which, due to a combination of small vertical scale, rapid rotation and strong stratification may be described by 2D Navier–Stokes or related equations.³⁰

Before discussing Kraichnan’s contributions, however, a brief digression on prior 2D turbulence work is in order. For many years, following the observation by Taylor that there cannot

²⁸ Kraichnan, 1958a, 1959b, Kawasaki, 1970.

²⁹ The first quote is from Kraichnan, 1975a: p. 31 and the second from Kraichnan & Chen, 1989: p. 162 ; see also Forster et al., 1977. For the simulation of truncated Euler, Cichowlas et al., 2005.

³⁰ The pioneering paper on 2D turbulent cascades is Kraichnan, 1967b.

be appreciable energy dissipation in high-Reynolds-number 2D flow, two-dimensional turbulence was not considered with much favor. Vortex stretching was regarded as the essence of turbulence and this effect is absent in 2D. Thus, it was often stated—and occasionally still is—that “there is no 2D turbulence.” This situation changed in the late forties, when two-dimensional approximations were proposed for high-Reynolds-number geophysical flows, which had many of the attributes of turbulence (randomness, disorder, etc.) For example, J.G. Charney in 1948 formulated the quasi-geostrophic model, in which potential vorticity (like vorticity for 2D Euler) is conserved along every fluid element. A little later, J. von Neumann made a number of interesting remarks on 2D turbulence, in particular that it is expected to have far less disorder than in 3D, precisely because of vorticity conservation; but this material remained unpublished for a long time. At the very end of his 1953 turbulence monograph, G.K. Batchelor proposed to investigate the spottiness “of the energy of high wave-number components” using 2D turbulence. He observed that in a 2D ideal flow the conservation of the integral of (one half of) the squared vorticity—Now called “enstrophy”, a term coined by C.E. Leith, which we shall use liberally—prevents energy from solely flowing to high wavenumbers: some energy has to be transferred also to smaller wavenumbers (larger scales). Batchelor also concurred with Onsager about the tendency of small vortices of the same sign to merge into larger vortices. In the same year, R. Fjørtoft showed that, because of the simultaneous conservation of energy and enstrophy, it is impossible for 2D dynamics to change the amplitudes of only two (Fourier) modes. Within a triad of modes, he showed that the change in energy for the member of the triad with intermediate wavenumber is the opposite to that of the other two members and that the member with lowest wavenumber shows the largest energy change of the two extreme members.³¹

It is thus clear that there must be significant inverse transfer of energy in 2D. However, even in 3D there is some inverse transfer of energy for the case of freely decaying high Reynolds number flow, where the peak of the energy spectrum migrates to smaller wavenumbers. What about the K41 energy cascade and the presence of an inertial range over which the energy flux is uniform? Lee showed that a *direct* energy cascade is not possible in 2D, because it would violate enstrophy conservation.³² Before 1967 no conjecture is found in the literature positing any type of 2D power-law cascade range.

Then comes Kraichnan. The Abstract of his first 1967 paper is worth quoting in full:

Two-dimensional turbulence has both kinetic energy and mean-square vorticity as inviscid constant of motion. Consequently it admits two formal inertial ranges, $E(k) \sim \epsilon^{2/3} k^{-5/3}$ and $E(k) \sim \eta^{2/3} k^{-3}$, where ϵ is the rate of cascade of kinetic energy per unit mass, η is the rate of cascade of mean-square vorticity, and the kinetic energy per unit mass is $\int_0^\infty E(k) dk$. The $-\frac{5}{3}$ range is found to entail backward energy cascade, from higher to lower wavenumbers k , together with zero-vorticity flow. The -3 range gives an upward vorticity flow and zero-energy flow. The paradox in these results is resolved by the irreducibly triangular nature of the elementary wavenumber interactions. The formal -3 range gives a nonlocal cascade and consequently must be modified by logarithmic factors. If energy is fed in at a constant rate to a band of wavenumbers $\sim k_i$ and the Reynolds number is large, it is conjectured that a quasi-steady-state results with a $-\frac{5}{3}$ range for $k \ll k_i$ and a -3 range for $k \gg k_i$, up to the viscous cutoff. The total kinetic energy increases steadily with time as the $-\frac{5}{3}$ range pushes to ever-lower k , until scales the size of the entire fluid are strongly excited. The rate of energy dissipation by viscosity decreases to zero if kinematic viscosity is decreased to zero with other parameters unchanged.

This is followed by a detailed and very dense presentation of the arguments supporting such results. Other than the works already cited and known to Kraichnan, there were no experimental or numerical results which could guide him. Furthermore, he took up the formidable challenge of not using closure, although he stated:

³¹ See Taylor, 1917: pp. 76-77, Charney, 1947; von Neumann, 1949: §§ 2.3-2.4; Batchelor, 1953: pp. 186-187; Fjørtoft, 1953. CROSS REFS. MAY BE NEEDED

³² Lee, 1951.

No use is made of closure approximations. However, the Lagrangian-history direct-interaction approximation, which yields Kolmogorov's similarity cascade in three dimensions, preserves the vorticity constraint in two dimensions and appears to yield the principal dynamical features inferred in the present paper.

Actually, some of the Fourier-space machinery developed by Kraichnan for closure, such as the use of the energy flux $\Pi(k)$ passing through wavenumber k and of the triad contributions to energy transfer $T(k, p, q)$, remains meaningful for homogeneous isotropic turbulence without recourse to closure. Of course, such quantities must then be expressed in terms of triple correlations of the velocity field and not solely in terms of the energy spectrum. Invoking a scaling ansatz for triple correlations, $T(ak, ap, aq) = a^{-(1+3n)/2}T(k, p, q)$, and also for the energy spectrum,³³ $E(k) \propto k^{-n}$, Kraichnan showed that for $n = 5/3$ one obtains an inertial range with zero enstrophy flux and non-vanishing energy flux and that for $n = 3$ one obtains an inertial range with zero energy flux and non-vanishing enstrophy flux.

These would be the famous dual cascades of 2D turbulence if Kraichnan could show that the former range has a negative energy flux, while the latter has a positive enstrophy flux. At this point, to predict the directions of the cascades, he made a very creative use of the equilibrium statistical mechanics of the Galerkin-truncated Euler equations, which he now calls "absolute (statistical) equilibria." In 2D, because of the simultaneous conservation of energy E and enstrophy Ω , it is easy to show that when the Euler equations are Galerkin truncated to a wavenumber band $[k_{\min}, k_{\max}]$ with $0 < k_{\min} < k_{\max}$, the absolute equilibrium $e^{-(\alpha E + \beta \Omega)}/Z$ is Gaussian with an energy spectrum

$$E(k) = \frac{k}{\alpha + \beta k^2}, \quad (13)$$

where α and $\beta > 0$ are constrained by the knowledge (say, from the initial conditions) of the total energy $E = \int_{k_{\min}}^{k_{\max}} E(k)dk$ and of the total enstrophy $\Omega = \int_{k_{\min}}^{k_{\max}} k^2 E(k)dk \geq k_{\min}^2 E$. If $\Omega/(k_{\min}^2 E)$ is very close to unity, it is seen that the "inverse temperature" α must be negative and that the spectrum (13) displays a strong peak near k_{\min} . The situation is in stark contrast to the case of 3D absolute equilibria: for 3D parity-invariant flow the absolute equilibrium energy spectrum is proportional to k^2 , as we have seen, and always peaks at the highest wavenumber. These results suggested to Kraichnan that "a tendency toward equilibrium in an actual physical flow should involve an upward flow of vorticity and, therefore, by the conservation laws, a downward flow of energy." For good measure, Kraichnan gave two independent arguments to justify the same conclusions. Adapting earlier considerations of Fjørtoft, he noted that these cascade directions follow if the transfer in each triad is "a statistically plausible spreading of the excitation in wavenumber: out of the middle wavenumber into the extremes." Finally, he showed that such diffusive spreading in wavenumber indeed develops instantaneously for an initial Gaussian statistical distribution, applying expressions of W.H. Reid and Ogura for the quasi-normal closure in 2D.³⁴

But are these formal similarity ranges physically realizable? Kraichnan next turned to this question. Since a $k^{-5/3}$ range gives a divergent energy at small wavenumbers and the k^{-3} range a logarithmically divergent enstrophy, cascade ranges of arbitrary extent require forcing at some intermediate wavenumber k_i . For finite ranges, Kraichnan noted, there must be some "leakage" of energy input ε to high wavenumbers and of enstrophy input η to low wavenumbers. As either of the ranges increases in length it becomes a "purer" cascade, due to the blocking effect of conservation of the dual invariant. The enstrophy cascade will proceed up to the cutoff wavenumber $k_d = (\eta/\nu^3)^{1/6}$ set by viscosity, with a vanishingly small energy dissipation $\varepsilon_d \sim \varepsilon(k_i/k_d)^2$ for $k_d \gg k_i$. If there is no minimum wavenumber k_0 , Kraichnan concluded that an inverse cascade should proceed for ever to smaller and smaller wavenumbers which, on dimensional grounds, scale as a $\varepsilon^{-1/2}t^{-3/2}$, where t is the time elapsed. These cascade ranges are only plausible universal states, Kraichnan observed, if

³³ As Kraichnan pointed out, the latter is not used in his argument, thus allowing, in present-day language, for anomalous scaling.

³⁴ Reid, 1959; Ogura, 1962.

the cascade dynamics are scale-local, with the dominant nonlinear interactions among triads of wavenumbers, all of comparable magnitude. Kraichnan concluded that the 2D inverse cascade $-5/3$ range is local just as is the 3D direct cascade $-5/3$ range of Kolmogorov. The 2D direct enstrophy range is at the margin of locality, however, and must have logarithmic corrections to exact self-similarity. Echoing the earlier ideas of von Neumann, Kraichnan argued that the infinite number of local vorticity invariants in 2D suggests non-universality of spectral coefficients. He then noted

A further point is that the nonlocalness of the transfer in the -3 range suggests in itself that [the] cascade there is not accompanied by degradation of the higher statistics in the fashion usually assumed in a three-dimensional Kolmogorov cascade. This is consistent with a picture of the transfer process as a clumping-together and coalescence of similarly signed vortices with the high-wavenumber excitation confined principally to thin and infrequent shear layers attached to the ever-larger eddies thus formed.

This is the only point in the paper where Kraichnan speculates on physical-space mechanisms, clearly influenced by the statistical mechanics argument of Onsager.

Kraichnan considered finally in his 1967 paper the situation that the fluid is confined to a finite domain with a minimum wavenumber $k_0 \ll k_i$. He wrote:

The conjecture is offered here that after the $-5/3$ range reaches down to wavenumbers $\sim k_0$ the downward cascade from k_i continues and the energy delivered to the bottom of the range piles up in the mode k_0 . As the energy in k_0 rises sufficiently, modification of the $-5/3$ range toward absolute equilibrium is expected, starting at the bottom and working up to progressively larger wavenumbers.

This conclusion was supported by considering, once again, the 2D absolute equilibria; for $\Omega/(Ek_0^2)$ close to unity, Kraichnan showed that they have nearly all of the energy carried by the “gravest” mode k_0 . He pointed out an analogy with quantum Bose condensation which, apparently, played a key role in stimulating his whole analysis. Indeed, he wrote in the Introduction:

The present study grew out of an investigation of the approach of a weakly coupled boson gas to equilibrium below the Bose-Einstein condensation temperature. There is a fairly close dynamical analogy in which the number density and kinetic-energy density of the bosons play the respective roles of kinetic-energy density and squared vorticity.

This quantum phenomenon was discussed at length in a prior Kraichnan paper in 1967.³⁵

Two very original concepts enter into turbulence theory with Kraichnan’s landmark work. The first idea is that a single system may support two, co-existing cascades with different spectral ranges, in 2D the dual cascades of energy and enstrophy. The second idea is there may be constant flux spectral ranges that correspond to an *inverse cascade*, from small to large scales. Kraichnan’s concept of the 2D inverse energy cascade is very far from the Richardson–Kolmogorov vision of 3D turbulence, in which energy, introduced at large scales either through the initial conditions or by suitable forces or by instabilities, cascades to smaller scales and eventually dissipates by viscosity into heat. Two other groups were, however, pursuing ideas very closely related to those of Kraichnan, at this same time. These were G.K. Batchelor and R.W. Bray at Cambridge, UK and V. Zakharov in the Soviet Union.

The 2D enstrophy cascade was proposed independently of Kraichnan, and even somewhat earlier, by Batchelor. This result is reported in the 1966 Cambridge PhD dissertation of Bray. At the beginning of § 1.4 he gave his supervisor, Batchelor, credit for the idea that the enstrophy dissipation rate could remain finite in the limit of vanishing viscosity. This led Bray to suggest an enstrophy cascade with a k^{-1} spectrum and thus a k^{-3} energy spectrum. Bray attempted to check this theory by performing a 2D spectral numerical

³⁵ Kraichnan, 1967a

simulation, probably the first of its kind. These results were not made public, however, until a 1969 paper authored by Batchelor. The analysis of Batchelor and Bray is remarkably complementary to Kraichnan's. They considered only decaying 2D turbulence, not forced steady states. Most of their physical discussion was also in real phase, not in spectral space, and focused on the analogy between stretching of vorticity-gradients in 2D and Taylor's vortex-stretching mechanism in 3D. Neither in Bray's thesis nor in Batchelor's paper was there any discussion of a separate $-5/3$ range in 2D with constant flux of energy to large scales (Batchelor was, by 1969, aware of Kraichnan's earlier paper and cited it in his work).³⁶

The notion of two distinct power-law ranges already appears in the 1966 PhD thesis of Zakharov, on the weak turbulence of gravity waves. One of these ranges was identified as a direct cascade of energy to high-wavenumber, but the physical interpretation of the other was not clearly identified in the thesis. Shortly afterward, however, Zakharov hit upon the idea that the second power-law range corresponds to an inverse cascade of wave action or "quasi-particle number." It thus appears that Kraichnan and Zakharov arrived independently at the idea of an inverse cascade although Kraichnan, it seems, got the idea slightly earlier into print. Both Kraichnan and Zakharov also clearly realized the applicability of these notions of dual cascades and inverse cascade to many other systems, including quantum dynamics of Bose condensation.³⁷

The subject of 2D turbulence continued to interest Kraichnan until the early 1980's, at least, and he wrote several later papers which sharpened the predictions and clarified the physics of his 1967 theory. In a 1971 *J. Fluid Mech.* article he applied his TFM closure to both 2D inverse and 3D direct energy cascades and obtained quantitative results on spectral coefficients. Kraichnan also studied the 2D enstrophy cascade and, in particular, worked out the logarithmic correction mentioned in 1967. A 1976 paper in *J. of Atmosph. Sci.* disseminated these results to the community of meteorologists, with special attention to the phenomenological concept of "eddy viscosity". Kraichnan proposed there a new interpretation of the inverse cascade in terms of a negative eddy viscosity, an idea that goes back to V. Starr and he gave a very simple heuristic explanation for this effect:

If a small-scale motion has the form of a compact blob of vorticity, or an assembly of uncorrelated blobs, a steady straining will eventually draw a typical blob out into an elongated shape, with corresponding thinning and increase of typical wavenumber. The typical result will be a decrease of the kinetic energy of the small-scale motion and a corresponding reinforcement of the straining field.

This idea has been particularly influential in the geophysical literature, where it has often been invoked to explain inverse energy cascade.³⁸

What is the empirical status of Kraichnan's dual cascade theory of 2D turbulence? A complete review would be out of place here, but we shall briefly discuss its verification with an emphasis on the most current work. Only very recently, in fact, has it become possible to observe both 2D cascades, inverse energy and direct enstrophy, in a single simulation. This has required herculean computations at spatial resolutions up to $32,768^2$ grid points. The simulations confirm Kraichnan's predictions for the $k^{-5/3}$ and k^{-3} ranges, with less accuracy for the latter due to finite-range effects. Earlier numerical simulations and laboratory experiments which have focussed on a single range have, however, separately confirmed the predictions of the 1967 paper. A number of numerical studies of the enstrophy cascade with "hyperviscosity" (powers of the Laplacian replacing the usual dissipative term) have reported observing the log-correction to the energy spectrum. The quasi-steady inverse cascade predicted by Kraichnan as a transient before energy from pumping reaches the largest scales has also been observed in both experiments and simulations, first by L. Smith

³⁶ Bray, 1966; Batchelor, 1969. CROSS REFS. MAY BE NEEDED

³⁷ Zakharov, 1966. B. Kadomtsev informed Zakharov of Kraichnan's 2D paper sometime around 1969 (V. Zakharov, private communication, 2010).

³⁸ For Kraichnan's application of TFM closure to 2D turbulence cascades, see Kraichnan, 1971b, 1976c; on negative viscosity, Starr, 1968; Kraichnan, 1976c; Kraichnan & Montgomery, 1980: §4.4; on vortex-thinning mechanism of inverse cascade, Kraichnan, 1976c; Rhines, 1979; Salmon, 1980, 1998: p. 229. Another notable paper on 2D turbulence is Kraichnan, 1975b.

and V. Yakhot. The constant flux $k^{-5/3}$ range is cleanly observed. However, contrary to Kraichnan’s speculations in his 1967 paper, the cascade is not associated to “coalescence” of vortices and, indeed, the statistics of the velocity are quite close to Gaussian and strong, coherent vortices do not appear until the energy begins to accumulate at the largest scales. Experiments and simulations on the statistical steady-state have instead found considerable evidence for Kraichnan’s 1976 “vortex-thinning” mechanism of energy transfer even in the local cascade regime. In the situation without large-scale damping, there is “energy condensation” at large scales as Kraichnan had supposed, but not confined to the gravest mode. Recent simulations show that condensation in a periodic domain appears as a pair of large, counterrotating vortices with a k^{-3} spectrum. These vortices are close to what is predicted by an equilibrium, maximum-entropy argument although the system is non-equilibrium, with continuously growing energy and constant negative energy flux.³⁹

Perhaps the most interesting question, from the general scientific point of view, is the relevance of Kraichnan’s ideas to planetary atmospheres and oceans. This question is complicated by the limited scale ranges that exist in those systems and the greater complexity of the dynamics. However, several recent observational studies have found evidence for both inverse energy and direct enstrophy cascades in the Earth’s atmosphere and oceans.⁴⁰

IV. INTERMITTENCY

Intermittency is a rather general term referring to the spottiness of small-scale turbulent activity, be it at dissipation-range scales or at inertial-range scales. In the late forties Batchelor and A.A. Townsend observed intermittent behavior of low-order velocity derivatives; since such derivatives come predominantly from the transition region between the inertial and dissipation ranges, this intermittency cannot be directly taken as evidence that the self-similarity postulated for the K41 inertial-range is breaking down.⁴¹

A. Dissipation-range intermittency

Kraichnan was the first to explain intermittency in the far dissipation range or, equivalently, for high-order velocity derivatives. In slightly modernized form, his argument is as follows: suppose that the flow can be divided into macroscopic regions each having its energy dissipation rate ε and its energy spectrum $E(k)$. In the far dissipation range $E(k)$ falls off faster than algebraically. From DIA results or from von Neumann’s analyticity conjecture regarding solutions of the Navier–Stokes equations, Kraichnan expects a fall-off $\propto \exp(-k/k_d)$, where the dissipation wavenumber k_d is given, at least approximately, by the K41 expression $(\varepsilon/\nu^3)^{1/4}$. When $k \gg k_d$, even minute macroscopic fluctuations in ε , which are very likely as pointed out by L.D. Landau, will produce huge macroscopic fluctuations in $E(k)$ and thus strong intermittency in physical-space filtered velocity signals obtained by keeping only those Fourier coefficients which are in a high- k -octave of wavenumbers. This argument can be made more systematic by using singularities of the analytic continuation of the velocity field to complex space-time locations.⁴²

³⁹ On numerical simulation of simultaneous cascades, see Boffetta, 2007, Boffetta & Musacchio, 2010; on the log-correction in the enstrophy cascade, Borue, 1993; Gotoh, 1998; Pasquero & Falkovich, 2002; on quasi-steady cascade, Smith & Yakhot, 1993; on vortex thinning, Chen et al., 2006; Xiao et al. 2008; on condensations, cf. Chertkov et al., 2007; Bouchet & Simonnet, 2009.

⁴⁰ On direct enstrophy cascade in the Earth’s stratosphere, Cho & Lindborg, 2001; on inverse energy cascade in the South Pacific, Scott & Wang, 2005.

⁴¹ Batchelor & Townsend, 1949. CROSS REFS. MAY BE NEEDED

⁴² Kraichnan, 1967c, 1974a: p. 327; von Neumann, 1949; on complex singularities, cf. Frisch & Morf, 1981.

B. Inertial-range intermittency

Much more difficult is the issue of intermittency at inertial-range scales and the problem of *anomalous scaling*, that is scaling for which the exponents cannot be obtained by a dimensional argument, as in K41. In the early sixties, A.M. Obukhov and his advisor Kolmogorov began to suspect that K41 must be somewhat modified because spatial averages ε_r of the local energy dissipation over balls with a radius r staying within the inertial range appeared to fluctuate more and more when r is decreased; they proposed a lognormal model of intermittency.⁴³ E.A. Novikov and R. Stewart and then A.M. Yaglom constructed ad hoc random multiplicative models to capture such intermittency and the corresponding scaling exponents. Mandelbrot showed that, in these models, the dissipation is taking place on a set with non-integer *fractal* dimension; in general such models are actually *multifractal*.⁴⁴

For Kraichnan, who liked proceeding in a systematic way, keeping as much contact as possible with the true fluid-dynamical equations, inertial-range intermittency was a very difficult problem. Indeed, it was known since 1966 that the full hierarchy of moment or cumulant equations derived for statistical solutions of the Navier–Stokes equation is compatible with the scale-invariant K41 theory in the limit of infinite Reynolds numbers. But Kraichnan was also aware that K41 is equally compatible with the Burgers equation, which definitely has no K41 scaling (because of the presence of shocks); he also noticed that the presence of the pressure in the incompressible Navier–Stokes was likely to reduce the intermittency one would otherwise expect from a simple vortex stretching argument. Closure seemed incapable to say anything about the breaking of the K41 scale invariance (one major exception to this statement is discussed in Section IV.C).⁴⁵

At first Kraichnan examined critically the toy models developed by the Russian school and observed that ε_r is not a pure inertial-range quantity and proposed to study intermittency in terms of more appropriate quantities, such as the local fluctuations of the energy flux associated with a wavenumber k in the inertial range. An estimate of this flux is u_r^3/r , where $r \sim 1/k$ and u_r is, say, the modulus of the velocity difference between two points separated by a distance r . With this in mind, he wrote:

If we increase the intermittency by making the fluid into quiescent regions with negligible velocity and active regions, of equal extent, where u_r increases by $\sqrt{2}$, then the mean kinetic energy in scales order r is unchanged but the time constant decreases, and hence ε increases, by $\sqrt{2}$. This example suggests, first, that if Kolmogorov’s theory holds in subregions of the fluid, then the constant $f(0)$ in the inertial-range law can be universal only if intermittency in the local dissipation ε_r , defined as average dissipation over a domain of size r , somewhat tends to a universal distribution. Second, if intermittency increases as scale size decreases, and Kolmogorov’s basic ideas hold in local regions, then the cascade becomes more efficient as r decreases and $E(k)$ must fall off more rapidly than $k^{-5/3}$ if, according to conservation of energy, the overall cascade rate is r independent.⁴⁶

A few years later this remark, together with ideas of Mandelbrot, became a key ingredient in the development of the β -model, a phenomenological model of intermittency that uses exclusively inertial-range quantities.⁴⁷

Kraichnan pursued some of these ideas further himself in an influential 1974 paper in J. Fluid Mech. This paper is pure Kraichnan. A wealth of intriguing ideas are tossed out, very original model calculations sketched in brief, and clever counterexamples devised against conventional ideas. At least two contributions of this paper are now well-known. First,

⁴³ See the contribution by Falkovich in this volume. CROSS REFERENCES TO OTHER PAPERS IN THIS VOLUME NEEDED.

⁴⁴ Obukhov, 1962; Kolmogorov, 1962; Novikov & Stewart, 1964; Yaglom, 1966; for a review of the Russian work, cf. Falkovich CROSS REFS. MAY BE NEEDED; Mandelbrot, 1968, 1974; on multifractality, cf. Parisi & Frisch, 1985.

⁴⁵ On K41-compatibility of the Navier–Stokes equations, cf. Orszag & Kruskal, 1966; on the differences between Burgers and Navier–Stokes turbulence, cf. Kraichnan, 1974a, 1991.

⁴⁶ Kraichnan, 1972: p. 213.

⁴⁷ On using inertial-range quantities, cf. Kraichnan, 1972: p. 213, 1974a; Frisch, Sulem & Nelkin, 1978.

Kraichnan proposed a refined similarity hypothesis (RSH) alternative to that of Kolmogorov, which he based on inertial-range energy flux rather than volume-averaged energy dissipation. Later numerical and experimental studies have confirmed Kraichnan’s RSH (and also that of Kolmogorov!). In the same paper, Kraichnan gave what is now the standard formulation of the “Landau argument” on intermittency and non-universality of coefficients in scaling laws. His argument is considerably clearer and more compelling than the brief remarks originally made by Landau in 1942. The crucial observation of Kraichnan is that only those K41 predictions which are linear in the ensemble-average energy dissipation $\langle \varepsilon \rangle$ —such as the Kolmogorov 4/5-th law—can be expected to be universally valid inertial-range laws. Other K41 predictions which depend upon fractional powers $\langle \varepsilon \rangle^p$ are not invariant to composition of sub-ensembles with distinct global values of mean-dissipation. There is at least as much Kraichnan in this argument as there is Landau.⁴⁸

C. Passive scalar intermittency and the “Kraichnan model”

The story of the Kraichnan model and of the birth of the first *ab initio* derivation of anomalous scaling is rather complex, spanning nearly three decades. Since it is understood that in this book the emphasis should be on what happened before 1980, we shall concentrate on the early developments, that begin in the late sixties.

The transport of a passive scalar field (say a temperature field $T(\mathbf{x}, t)$), advected by a prescribed incompressible turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ and subject to molecular diffusion with a diffusivity κ , is governed by the following *linear* stochastic equation:

$$[\partial_t + \mathbf{u}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} - \kappa \nabla_{\mathbf{x}}^2] T(\mathbf{x}, t) = 0. \quad (14)$$

The qualification “passive” is used when there is no or negligible back-reaction on the turbulent flow of the field being transported. Examples of passive scalar transported fields are provided by the temperature of a fluid (when buoyancy is negligible), the humidity of the atmosphere, the concentration of chemical or biological species. Passive scalar transport has thus an important domain of applications and considerable efforts were made since the forties to gain an understanding at least as good as for turbulence dynamics. In particular Obukhov and, independently, S. Corrsin derived for passive scalars the counterpart of the $-5/3$ law, Yaglom derived an analogue of the Kolmogorov’s four-fifths law and Batchelor derived a k^{-1} passive scalar energy spectrum in a regime of fully developed turbulence with large Schmidt number (see below). It was thus quite natural for Kraichnan to see how well the closure tools he developed for turbulence in the fifties and the sixties were able to cope with passive scalar dynamics. He applied his LHDIA closure to the passive scalar problem, for example, reproducing Obukhov–Corrsin scaling with precise numerical coefficients.⁴⁹

In 1968 Kraichnan realized that a closed equation can be obtained for a scalar field passively advected by a turbulent velocity with a very short correlation time, without any further approximation. The DIA closure is exact for this special system, reducing to a single equation for the scalar correlation function at two space points and simultaneous times. The mean Green function reduces to a Dirac delta because of the zero correlation-time assumption. This 1968 model is now usually called “the Kraichnan model” [of passive scalar dynamics] and has assumed a paradigmatic status for turbulence theory, comparable to that of the Ising model in statistical mechanics of critical phenomena. Its importance stems from a string of major discoveries by Kraichnan and others on the fundamental mechanism of intermittency, some of which will be described only briefly because they took place in the nineties. Kraichnan showed that even when the velocity field is not at all intermittent, e.g. a Gaussian random field, the passive scalar (henceforth called “temperature” for brevity) can become intermittent and this in several ways.⁵⁰

⁴⁸ For Landau’s argument, Landau & Lifshitz, 1987; Kraichnan, 1974a: § 2; Frisch, 1995: § 6.4.

⁴⁹ Obukhov, 1949; Corrsin, 1951; Yaglom, 1949; Batchelor, 1959; cf. also the chapters on Batchelor, on Corrsin and on the Russian School in this volume. CROSS REFS. MAY BE NEEDED. On LHDIA for passive scalars, cf. Kraichnan, 1965a: § 5–7.

⁵⁰ Kraichnan, 1968b, 1974b, 1994.

A first mechanism, which applies in the far dissipation range, is basically the same as described in §IV.A and will not concern us further.

A second mechanism identified by Kraichnan concerns the so-called Batchelor regime: when the Schmidt number ν/κ is large, there is a range of scales for which the velocity field is strongly affected by viscous dissipation, but the temperature field does not undergo much diffusion; in this regime the velocity field can be locally replaced by a uniform random shear.⁵¹ Tiny, well-separated temperature blobs are then stretched and squeezed in a way which is amenable to asymptotic analysis at large times. Actually doing this in a systematic way would have required all kinds of heavy-duty theoretical tools: path integrals, large deviation theory, fluctuations of Lyapunov exponents, etc.⁵² It is then possible to show that the distributions of spatial derivatives of the temperature display a lognormal-type intermittency at zero diffusivity⁵³ and a weaker form of intermittency in the regime with non-zero diffusivity. Actually, all this was done — and correctly so — by Kraichnan in a remarkable paper published in 1974, just after the paper on Kolmogorov’s inertial-range theories.⁵⁴ This paper is a tour de force, combining very original analytical arguments and deep physical intuition to reach exact conclusions, without any assistance from the advanced mathematical methods that were later applied to this problem. Kraichnan’s analysis was carried out for general space dimension d — following a suggestion of M. Nelkin—and one intriguing finding was that intermittency of the scalar vanished in the limit $d \rightarrow \infty$. Kraichnan’s work, which was going to strongly influence subsequent more formally rigorous analyses, showed a thorough understanding of the mechanism of intermittency in the Batchelor regime.

The third mechanism identified by Kraichnan was rather close to one of the Holy Grails of turbulence theory, namely understanding inertial-range anomalous scaling and predicting the scaling exponents. In 1994 Kraichnan conjectured that when the velocity $\mathbf{u}(\mathbf{x}, t)$ is Gaussian with a power-law spectrum (K41 would be one instance) and with a very short correlation time (white-in-time), then for vanishingly small κ the structure functions of the temperature display anomalous scaling. This is a rather amazing proposal: how can a self-similar velocity field act on a transported temperature field to endow it with anomalous scaling and thus with lack of self-similarity? As we shall see, the qualitative aspects of Kraichnan’s conjecture have been fully corroborated by later work.⁵⁵

Now we shall have to become slightly more technical to explain how Kraichnan tackled this problem, starting with his 1968 work. Let us rewrite the temperature equation (14) in abstract form

$$\partial_t T(t) = MT(t) + \tilde{M}(t)T(t), \quad (15)$$

where M is a linear deterministic operator (diffusion) and $\tilde{M}(t)$ a linear random operator (advection) with vanishing mean and “very short correlation time”. More precisely, one performs the substitution

$$\tilde{M}(t) \longrightarrow \frac{1}{\epsilon} \tilde{M}\left(\frac{t}{\epsilon^2}\right), \quad \epsilon \rightarrow 0, \quad (16)$$

where $\tilde{M}(t)$ is statistically stationary. In the limit $\epsilon \rightarrow 0$ the temperature becomes a Markov process (in the time variable) and it may be shown that the mean temperature satisfies a *closed* equation, namely⁵⁶

$$\partial_t \langle T \rangle = M \langle T(t) \rangle + \mathcal{D} \langle T(t) \rangle, \quad (17)$$

$$\mathcal{D} = \int_0^\infty \langle \tilde{M}(s) \tilde{M}(0) \rangle ds. \quad (18)$$

⁵¹ Batchelor, 1959 and Moffatt’s contribution in this volume. CROSS REFS. MAY BE NEEDED

⁵² See, e.g., the review by Falkovich, Gawędzki & Vergassola, 2001.

⁵³ This is a non-trivial variant of the obvious result that when $m(t)$ is a scalar Gaussian random function the solution of the differential equation $dq(t)/dt = m(t)q(t)$ with $q(0) = 1$ is lognormal.

⁵⁴ Kraichnan, 1974b.

⁵⁵ Kraichnan, 1994.

⁵⁶ Hashminkii, 1966, Frisch & Wirth, 1997.

Similar closed equations can be derived for p -point moments of the temperature. In 1968 Kraichnan derived the equation for the two-point temperature correlation functions by this technique and found that the second-order temperature structure functions displayed scaling. The scaling exponent ζ_2 can actually be obtained by simple dimensional analysis. So far no evidence of anomalous scaling had emerged.

By a method similar to that used in 1968 for the two-point correlations of a passive scalar, Kraichnan derived in 1994 an equation for the structure function of order p . This equation is not closed (contrary to the equation for the p -point correlation function), but Kraichnan proposed a plausible approximate closure ansatz from which he derived the following scaling exponents ζ_p for the p th order structure function:

$$\zeta_{2p} = \frac{1}{2} \sqrt{4pd\zeta_2 - 2 + (d - \zeta_2)^2} - \frac{1}{2}(d - \zeta_2). \quad (19)$$

Since ζ_{2p} is obviously not equal to $p\zeta_2$, as would be required by self-similarity, (19) implies anomalous scaling. One year later it was shown that there is indeed anomalous scaling, using a *zero modes* method, borrowed partially from field theory: the equation for the moments of order $2p$ has a linear operator L_{2p} acting on the $2p$ -point correlation function and an inhomogeneous right hand side involving correlation functions of lower order. The zero modes correspond to certain functions of $2p$ variables which are killed by L_{2p} . Actually, determining the zero modes turned out to be quite difficult. In most instances it could be done only perturbatively, using as small parameter either the roughness exponent ξ of the prescribed velocity field or the inverse of the dimension of space d (as anticipated by Kraichnan's 1974 paper). The results agreed with numerical simulations, but did not agree with (19) except for a single value $\xi \doteq 1$. Kraichnan's prediction (19) must cross the numerical curve at one point, trivially, but it is possibly significant that Kraichnan's closure ansatz works best in the regime where the cascade dynamics is scale-local.⁵⁷

V. MISCELLANY AND CONCLUSIONS

A. Scattering of sound by turbulence

In 1952 M.J. Lighthill published a landmark paper on the generation of sound by turbulence. The next year Kraichnan observed that the production of noise in this theory depends on a high power of the Mach number and that the “*scattering* [of sound by turbulence] is the most conspicuous acoustical phenomenon associated with very low Mach number turbulence.” In his very first published paper, Kraichnan developed a systematic theory of the interaction of sound with nearly incompressible turbulence. This paper, together with further developments was to be the basis of a nonintrusive ultrasonic technique for the remote probing of vorticity. The same year 1953 and independently Lighthill also published a theory of scattering.⁵⁸

In his approach to the problem of interaction of sound and turbulence, Kraichnan assumed that the turbulence is incompressible and can be described by a divergence-free velocity, whereas the sound is given by a curl-free (potential) velocity. As done by Lighthill, Kraichnan assumed that density and pressure fluctuations are related by an adiabatic equation of state with a uniform speed of sound. Starting with the full compressible equations he performed a decomposition of the velocity

$$\mathbf{u} = \mathbf{u}^L + \mathbf{u}^T \quad (20)$$

⁵⁷ Kraichnan, 1994. On zero-mode methods, cf. Gawędzki & Kupiainen, 1995; Chertkov et al., 1995; Shraiman & Siggia, 1995 and the review by Falkovich, Gawędzki & Vergassola, 2001. On simulations, cf. Frisch, Mazzino & Vergassola, 1998; Gat, Procaccia & Zeitak, 1998; Frisch et al. 1999. The whole story about anomalous scaling for passive scalars is recounted in www.ocean.eu/etc7/work-on-passive-scalar.pdf.

⁵⁸ Lighthill, 1952, 1953; Kraichnan, 1953; for further developments, see, e.g., Lund & Rojas, 1989; Ting & Miksis, 1990; for vorticity probing, see, e.g. Baudet, Ciliberto & Pinton, 1991.

into a curl-free (longitudinal in the spatial Fourier space) and a divergence-free (transverse in Fourier space) part. (This is known as a Hodge decomposition in mathematics.) He then obtained a wave equation which has four terms. One term is linear in \mathbf{u}^L , related to viscous stresses and is mostly negligible. The three remaining ones are quadratic and of type L-L, L-T and T-T. The T-T term is Lighthill's (quadrupolar) sound production term. The L-T term gives the scattering of a preexisting sound wave by the turbulence. Kraichnan then worked out the angular distribution and frequency distribution of the scattered wave in terms of the four-dimensional Fourier transform of the shear velocity field. Explicit expressions for cross sections were obtained for the case of a scattering from a region of isotropic turbulence.

Some remarks are in order. Kraichnan worked in relativity and quantum field-theory for several years before engaging in hydrodynamics but this first published paper is about hydrodynamics;⁵⁹ the turbulent flow is here prescribed and defined as "characterized by the fact that although the detailed structure of the system is not known, suitable averages of certain quantities are known for a representative ensemble of similar systems." The paper is unusually well written for a first paper and indicates considerable maturity of the young scientist who had already been active for six years, although he refrained from publishing.

B. High-Rayleigh number convection

Thermal convection is ubiquitous in technology and is amenable to controlled experiments where a fluid heated from below is placed between two horizontal plates. Within the so-called Boussinesq approximation, the dimensionless parameters are the Rayleigh number $Ra \equiv g\alpha\delta T h^3/(\nu\kappa)$ and the Prandtl number $Pr \equiv \nu/\kappa$. Here, g is the acceleration due to gravity, δT the vertical temperature difference across the fluid of height h , and α , ν and κ are the thermal expansion coefficient of the fluid, its kinematic viscosity and its thermal diffusivity. Turbulent thermal convection was and remains a central topic of the Woods Hole Oceanographic Institute Geophysical Fluid Dynamics summer program, with which Kraichnan had considerable interaction from the late fifties. Around the same time he also had much interaction with E.A. Spiegel, who had been trained in astrophysical fluid dynamics: it is usually convective transport which allows the heat generated in the interiors of stars to escape. In the early days the easiest way to model astrophysical convection was through the mixing length theory, which follows ideas of Boussinesq and of Prandtl. In 1962 Kraichnan devoted a fairly substantial paper to thermal convection, which we cannot summarize in detail because of lack of space. We shall thus concentrate on his most original contribution, to what is now called "ultimate convection", at extremely high Rayleigh numbers.⁶⁰

One important question in high-Rayleigh number convection is the dependence upon Rayleigh and Prandtl numbers of the Nusselt number N , the heat flux non-dimensionalized by the conductive heat flux. In the fifties C.H.B. Priestley found a dimensional argument which suggests that for high Rayleigh numbers $N \propto Ra^{1/3}$. As pointed out by Kraichnan "[In] Priestley's theory ... it is assumed that at sufficiently high Rayleigh numbers most of the change in mean temperature across the layer occurs in thin boundary regions, at the surfaces, where molecular heat conduction and molecular viscosity are dominant. Elsewhere ... convective heat transport and eddy viscosity are dominant."⁶¹

However, at sufficiently high Rayleigh numbers the thermal boundary layer may be destroyed and another regime may emerge, which, as shown by Kraichnan, has an approximately square-root dependence on Ra . This can be partially derived by a simple dimensional argument due to Spiegel, which assumes that the heat flux depends neither on the viscosity nor on the thermal diffusivity and which gives $N \sim (RaPr)^{1/2}$.⁶² Kraichnan's derivation makes use of the phenomenological theory of high-Reynolds number shear flow turbulence near a solid boundary, which gives a logarithmic correction proportional to $(\ln Ra)^{-3/2}$. Kraichnan also discussed the Prandtl number dependence of the various regimes. This al-

⁵⁹ His first relativity paper was to be published only two years later (Kraichnan, 1955b).

⁶⁰ Boussinesq, 1870; Prandtl, 1925; Kraichnan, 1962.

⁶¹ Priestley, 1959 and references therein; Kraichnan, 1962: p. 1374.

⁶² It may be shown that this argument breaks down at large Prandtl numbers.

lowed him to predict how high the Rayleigh number should be for the square-root regime to dominate: for a unit Prandtl number this threshold is around $Ra = 10^{21}$. It is generally believed that the threshold is significantly lower and depends on the Prandtl number and the boundary conditions. Successful attempts to observe this law may have been made with Helium gas. Artefacts masquerading as a $Ra^{1/2}$ law cannot be ruled out. In Göttingen a two-meter high convection experiment using sulfur hexafluoride (SF6) at 20 times atmospheric pressure is under construction to try and capture Kraichnan’s ultimate convection regime.⁶³

C. Kraichnan and computers

Kraichnan, although basically a theoretician, was very far from being allergic to computers. Actually, not only was he a very talented programmer, but he got occasionally involved in writing system software and even in modifying hardware. Some of his closest collaborators, foremost S.A. Orszag, prodded by him, got deeply involved in three-dimensional simulations of Navier–Stokes turbulence. This was — and still is — called “Direct Numerical Simulation” (DNS) because the original goal was to check on the validity of various closures by going *directly* to the fluid dynamical equations. Convinced that many features of high-Reynolds number turbulence should be universal, Kraichnan encouraged the use of the simplest type of boundary conditions (periodic) which allows the simple and efficient use of spectral methods. He also suggested using Gaussian initial conditions rather than more realistic ones. Curiously, although the thrust to do DNS started just after Kraichnan’s discovery of the 2D inverse cascade, he strongly recommended focusing on 3D flow.

Considerable effort — this time often in collaboration with J.R. Herring — went into the numerical integration of various closure equations. Kraichnan proposed using discrete wavenumbers in geometric rather than arithmetic progression. This allowed reaching very high Reynolds numbers. Kraichnan himself was actively involved in writing code for these investigations. His punched cards were shipped from New Hampshire to NASA Goddard Institute where the machine computations were performed during the 1960’s and 1970’s. Herring recalls that “Bob’s programs very rarely contained any bugs.”

D. Conclusions

Our survey has focussed on three of Kraichnan’s contributions to turbulence theory: (1) spectral closures and realizability, (2) inverse cascade of energy in 2D turbulence and (3) intermittency of passive scalars advected by turbulence. These are, arguably, his most significant achievements which have had the greatest impact on the field. Spectral closures of the DIA class still have numerous interesting applications when the questions under investigation do not depend crucially on deviations from K41. Even today an EDQNM calculation, for example, will often be the first line of assault on a difficult new turbulence problem. Furthermore, Kraichnan’s criterion of realizability has become part of the standard toolbox of turbulence closure techniques. Realizability is necessary both for physical meaningfulness and, often, for successful numerical solution of the closure equations. Kraichnan’s prediction of inverse cascade has been well verified by experiments and simulations and has relevance in explaining dynamical processes in the Earth’s atmosphere and oceans. The concept of an inverse cascade has proved very fruitful in other systems, too, where similar fluxes of invariants to large-scales may occur, such as magnetic helicity in 3D MHD turbulence, magnetic potential in 2D MHD turbulence, and particle number in quantum Bose systems. Finally, Kraichnan’s model of a passive scalar advected by a white-in-time Gaussian random velocity has become a paradigm for turbulence intermittency and anomalous scaling—an “Ising model” of turbulence. The theory of passive scalar intermittency has not yet led to a similar

⁶³ Spiegel, 1971. On artefacts, Sreenivasan (private communication, 2010). On Helium gas experiments, cf. Chavanne et al., 2001; Niemela et al., 2000. On SF6, cf. <http://www.sciencedaily.com/releases/2009/12/091203101418.htm>

successful theory of intermittency in Navier–Stokes turbulence. However, the Kraichnan model has raised the scientific level of discourse in the field by providing a nontrivial example of a multifractal field generated by a turbulence dynamics. It is no longer debatable that anomalous scaling is *possible* for Navier–Stokes.⁶⁴

A review of Kraichnan’s scientific legacy within the length constraint of this book must be very selective. For example we have not been able to discuss the numerous interactions Kraichnan had with many people in the USA and in other countries, particularly in France, Israel and Japan. Even focusing our discussion to his turbulence research prior to 1980, we have been forced to omit mention of a large number of problems to which Kraichnan made important contributions in that period. Furthermore, Kraichnan invested substantial time to other theoretical approaches, that came after the 1980 cutoff for this paper. We provide below just a few references to this additional work on turbulence by Kraichnan.⁶⁵ It may be that later generations will find that our survey has missed some of Kraichnan’s most significant accomplishments. The richness of his œuvre can only be appreciated by poring over his densely written research articles, bristling with original ideas and novel methods, for oneself. The reader who does so will be generously rewarded for his effort.

It is amusing to wonder what might be Einstein’s assessment (from the welkins) of his former assistant. He would probably have to conclude that Kraichnan had a lot of *Sitzfleisch*.⁶⁶ Kraichnan’s papers, numbering more than a hundred and spanning five decades, many of them formidably technical, bear ample witness to their author’s iron determination and staying power. Turbulence is a dauntingly difficult subject where any significant advance is won by a hard-fought battle; and yet Kraichnan has left his record of victories throughout the field. Several international conferences held in his honor are testimony to the lasting impact of Robert H. Kraichnan.⁶⁷

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References

- Abe, R. 1973. Expansion of a critical exponent in inverse powers of spin dimensionality, *Prog. Theor. Physics* **49**, 113–128.
 Alfvén, H. 1942. Existence of electromagnetic-hydrodynamic waves, *Nature* **150**, 405–406.
 Batchelor, G.K. 1950. On the spontaneous magnetic field in a conducting liquid in turbulent motion, *Proc. Roy. Soc. Lond. Ser. A* **201**, 405–416.

⁶⁴ On the inverse magnetic helicity, Frisch et al., 1975. On the inverse magnetic potential cascade, cf. Fyfe & Montgomery, 1976. On Bose condensates, cf. Semikoz & Tkachev, 1997. Regarding intermittency/anomalous scaling, note that there have been many incorrect “proofs” of their absence in Navier–Stokes turbulence, to which the Kraichnan model is a counterexample; cf., e.g., Belinicher & L’vov, 1987.

⁶⁵ On the various topics that could not be covered in this review, see for pressure fluctuations: Kraichnan, 1956a, 1956b, 1957a; shear flow turbulence: Kraichnan, 1964a; magnetic dynamo: Kraichnan & Nagarajan, 1967; Kraichnan, 1976a, 1976c, 1979b; Vlasov plasma turbulence: Kraichnan & Orszag, 1967c; predictability and error growth: Kraichnan, 1970b; Kraichnan & Leith, 1972; Burgers: Kraichnan, 1968a, 1999; Kraichnan & Gotoh, 1993; quantum turbulence: Kraichnan, 1967a; path-integrals: Kraichnan, 1958a: § 4.3; Lewis & Kraichnan, 1962; self-consistent Langevin models: Kraichnan, 1970c; variational approaches: Kraichnan, 1958a: § 4.3; Kraichnan, 1979a; Wiener chaos expansions: Kraichnan, 1979a; Padé approximants: Kraichnan, 1968c, 1970a; decimation: Kraichnan, 1985, 1988; mapping closure: Kraichnan et al., 1989, Kraichnan, 1991; Kraichnan & Kimura, 1993; Kraichnan & Gotoh, 1993; critique of Tsallis statistics for turbulence: Gotoh & Kraichnan, 2004.

⁶⁶ The Germans have aptly called *Sitzfleisch* the ability to spend endless hours at a desk doing grueling work. *Sitzfleisch* is considered by mathematicians to be a better gauge of success than any of the attractive definitions of talent with which psychologists regale us from time to time (Gian-Carlo Rota).

⁶⁷ Los Alamos (May 1998) for Kraichnan’s 70th birthday and Santa Fe (May 2009) and Beijing (September 2009) after he left us in 2008.

- Batchelor, G.K. 1953. *The Theory of Homogeneous Turbulence*, Cambridge University Press, Cambridge, UK.
- Batchelor, G.K. 1959. Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. General discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113–133.
- Batchelor, G.K. 1969. Computation of the energy spectrum in homogeneous two-dimensional turbulence, *Phys. Fluids Suppl. II* **12**, 233–239.
- Batchelor, G.K. & Townsend, A.A. 1949. The nature of turbulent motion at large wave-numbers. *Proc. R. Soc. Lond. A* **199**, 238–255.
- Baudet, C., Ciliberto, S. & Pinton, J.F. 1991. Spectral analysis of the von Kármán flow using ultrasound scattering *Phys. Rev. Lett.* **67**, 193–195.
- Belinicher, V.I. & L’vov, V.S. 1987. The scale-invariant theory of developed hydrodynamic turbulence *Zhurn. Eksp. Teor. Fiz.* **93**, 533–551.
- Boffetta, G. 2007. Energy and enstrophy fluxes in the double cascade of two-dimensional turbulence, *J. Fluid Mech.* **589**, 253–260.
- Boffetta, G. & Musacchio, S. 2010. An update of the double cascade in two-dimensional turbulence, *Phys. Rev. E* **82**, 016307.
- Borue, V. 1993. Spectral exponents of enstrophy cascade in stationary two-dimensional homogeneous turbulence, *Phys. Rev. Lett.* **71**, 3967–3970.
- Bouchet, F. & Simonnet, E. 2009. Random changes of flow topology in two-dimensional and geophysical turbulence, *Phys. Rev. Lett.* **102**, 094504.
- Boussinesq, J. V. 1870. Essai théorique sur les lois trouvées expérimentalement par M. Bazin pour l’écoulement uniforme de l’eau dans les canaux découverts, *C. R. Acad. Sci. Paris* **71**, 389–393.
- Bray, R.W. 1966. A study of turbulence and convection using Fourier and numerical analysis, PhD dissertation available in the library of the Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge [all chapters but the last on thermal convection are also available at <http://www.oca.eu/etc7/bray-phd1966.pdf>].
- Chandrasekhar, S. 1955. A theory of turbulence, *Proc. R. Soc. Lond.* **A229**, 1–19.
- Charney, J.G. 1947. The dynamics of long waves in a baroclinic westerly current, *J. Meteor.* **4**, 135–163.
- Chavanne, X., Chillà, F., Chabaud, B., Castaing, B. & Hébral, B. 2001. Turbulent Rayleigh-Bénard convection in gaseous and liquid He, *Phys. Fluids* **13**, 1300–1320.
- Chen, S., Ecke, R.E., Eyink, G.L., Rivera, M., Wan, M.-P., & Xiao, Z. 2006. Physical mechanism of the two-dimensional inverse energy cascade, *Phys. Rev. Lett.* **96**, 084502.
- Chertkov, M., Connaughton, C., Kolokolov, I. & Lebedev, V. 2007. Dynamics of energy condensation in two-dimensional turbulence. *Phys. Rev. Lett.* **99** 084501.
- Chertkov, M., Falkovich, G., Kolokolov, I. & Lebedev, V. 1995. Normal and anomalous scaling of the fourth-order correlation function of a randomly advected passive scalar, *Phys. Rev. E* **52**, 4924–4941.
- Cho, J.Y.N. & Lindborg, E. 2001. Horizontal velocity structure functions in the upper troposphere and lower stratosphere 1. Observations. *J. Geophys. Res.* **106**, 10223–32.
- Cichowlas, C., Bonaïti, C., Debbasch, F. & Brachet, M. 2005. Effective dissipation and turbulence in spectrally truncated Euler flows, *Phys. Rev. Lett.* **95**, 264502.
- Corrsin, S. 1951. On the spectrum of isotropic temperature fluctuations in an isotropic turbulence, *J. Appl. Phys.* **22**, 469–473.
- Donzis, D.A. & Sreenivasan, K.R. 2010. The bottleneck effect and the Kolmogorov constant in isotropic turbulence, *J. Fluid Mech.* **657**, 171–188.
- Falkovich, G., Gawędzki, K. & Vergassola, M. 2001. Particles and fields in fluid turbulence, *Rev. Mod. Phys.* **73**, 913–975.
- Fjørtoft, R. 1953. On the changes in the spectral distribution of kinetic energy for twodimensional nondivergent flow, *Tellus* **5**, 225–230.
- Forster D., Nelson, D.R. & Stephen, M.J. 1977. Large-distance and long-time properties of a randomly stirred fluid, *Phys. Rev. A* **16**, 732–749.
- di Francesco, P., Ginsparg, P. & Zinn-Justin, J. 1995. 2D gravity and random matrices, *Phys. Rep.* **254**, 1–133.
- Frisch, U. 1995. *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge, UK.
- Frisch U. & Burrett, R. 1970. Parastochastics. *J. Math. Phys.* **11**, 364–390.
- Frisch, U., Lesieur, M. & Brissaud, A. 1974. A Markovian random coupling model for turbulence, *J. Fluid Mech.* **65**, 145–152.
- Frisch, U., Mazzino, A. & Vergassola, M. 1998. Intermittency in passive scalar advection. *Phys. Rev. Lett.* **80**, 5532–5537.
- Frisch, U., Mazzino, A., Noullez, A. & Vergassola, M. 1999. Lagrangian method for multiple correlations in passive scalar advection. *Phys. Fluids* **11**, 2178–2186.
- Frisch, U. & Morf, R. 1981. Intermittency in nonlinear dynamics and singularities at complex

- times, *Phys. Rev. A* **23**, 2673–2705.
- Frisch, U., Pouquet, A., Leorat, J. & Mazure, A. 1975. Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *J. Fluid Mech.* **68**, 769–778.
- Frisch, U., Sulem, P.-L. & Nelkin, M. 1978. A simple dynamical model of intermittent fully developed turbulence, *J. Fluid Mech.* **87**, 719–736.
- Frisch, U. & Wirth, A. 1996. Inertial-diffusive range for a passive scalar advected by a white-in-time velocity field, *Europhys. Lett.* **35**, 683–687.
- Fyfe, D. & Montgomery, D. 1976. High-beta turbulence in two-dimensional magnetohydrodynamics *J. Plasma Phys.* **16**, 181–191.
- Gat, O., Procaccia, I. & Zeitak, R. 1998. Anomalous scaling in passive scalar advection: Monte Carlo Lagrangian trajectories, *Phys. Rev. Lett.* **80**, 5536–5539.
- Gawędzki, K. & Kupiainen, A. 1995. Anomalous scaling of the passive scalar, *Phys. Rev. Lett.* **75**, 3834–3837.
- Galtier, S., Nazarenko, S.V., Newell, A.C. & Pouquet, A. 2000. A weak turbulence theory for incompressible magnetohydrodynamics, *J. Plasma Phys.* **63**, 447–488.
- Goldreich, P. & Sridhar, S. 1997. Magnetohydrodynamic turbulence revisited, *Astrophys. J.* **485**, 680–688.
- Gotoh, T. 1998. Energy spectrum in the inertial and dissipation ranges of two-dimensional steady turbulence, *Phys. Rev. E* **57**, 2984–2991.
- Gotoh, T. & Kraichnan, R. H. 2004. Turbulence and Tsallis statistics. *Physica D* **193** 231–244.
- Grant, H.L., Stewart, R.W. & Moilliet, A. 1962. Turbulent spectra from a tidal channel, *J. Fluid Mech.* **12**, 241–268.
- Hashminskii, R.Z. 1966. A limit theorem for the solutions of differential equations with random righthand sides, *Theor. Prob. Appl.* **11**, 390–406.
- Herring, J.R. & Kraichnan, R.H. 1972. Comparison of some approximations for isotropic turbulence, in: *Statistical Models and Turbulence*, Eds. Rosenblatt, M. & Van Atta, C. Springer-Verlag, New York, pp. 148–194.
- Hopf, E. 1952. Statistical hydromechanics and functional calculus, *J. Rat. Mech. Anal.* **1**, 87–123.
- ’t Hooft, G. 1974. A planar diagram theory for strong interactions. *Nucl. Phys. B* **72**, 461–473.
- Iroshnikov, P.S. 1963. Turbulence of a conducting fluid in a strong magnetic field, *Astronomicheskii Zhurnal* **40**, No. 4, 742–750 in Russian); translated in *Soviet Astronomy* **7**, 566–571 (1964). [The Russian version has the correct initial “R” for his given name “Ruslan”; this was incorrectly translated as a “P”.]
- Kawasaki, K. 1970. Kinetic equations and time correlation functions of critical fluctuations, *Ann. Phys. (N.Y.)* **61**, 1–56.
- Kimura, Y. and Kraichnan, R.H. 1993. Statistics of an advected passive scalar. *Phys. Fluids A* **5**, 2264–2277.
- Kit, L.G. & Tsinober, A.B. 1971. Possibility of creating and investigating two-dimensional turbulence in a strong magnetic field. *Magnitnaya Gidrodinamika*, **7**, 27–34.
- Kolmogorov, A.N. 1941. The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Dokl. Akad. Nauk SSSR* **30**, 299–303.
- Kolmogorov, A.N. 1962. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number, *J. Fluid Mech.* **13**, 82–85.
- Kraichnan, R.H. 1953. Scattering of sound in a turbulent medium, *J. Acoust. Soc. Am.* **25**, 1096–1104.
- Kraichnan, R.H. 1955a. Statistical mechanics of an adiabatically compressible fluid, *J. Acoust. Soc. Am.* **27**, 438–441.
- Kraichnan, R.H. 1955b. Special-relativistic derivation of generally covariant gravitation theory, *Phys. Rev.* **98**, 1118–1122.
- Kraichnan, R.H. 1956a. Pressure field within homogeneous anisotropic turbulence. *J. Acoust. Soc. Am.* **28**, 64–72.
- Kraichnan, R.H. 1956b. Pressure fluctuations in turbulent flow over a flat plate. *J. Acoust. Soc. Am.* **28**, 378–390.
- Kraichnan, R.H. 1957a. Noise transmission from boundary-layer pressure fluctuations. *J. Acoust. Soc. Am.* **29**, 65–80.
- Kraichnan, R.H. 1957b. Relation of fourth-order to second-order moments in stationary isotropic turbulence, *Phys. Rev.* **107**, 1385–1490.
- Kraichnan, R.H. 1958a. Irreversible statistical mechanics of incompressible hydromagnetic turbulence, *Phys. Rev.* **109**, 1407–1422.
- Kraichnan, R.H. 1958b. Higher order interactions in homogeneous turbulence theory, *Phys. Fluids* **1**, 358–359.
- Kraichnan, R.H. 1958c. A theory of turbulence dynamics, in *Second Symposium on Naval Hydrodynamics*, Ref. ACR-38, Office of Naval Research, Washington, DC, pp. 29–44.
- Kraichnan, R.H. 1959a. Structure of isotropic turbulence at very high Reynolds numbers, *J. Fluid*

- Mech.* **5**, 497–543.
- Kraichnan, R. H. 1959b. Classical fluctuation-relaxation theorem, *Phys. Rev.* **113**, 1181–1182.
- Kraichnan, R.H. 1961. Dynamics of nonlinear stochastic systems, *J. Math. Phys.* **2**, 124–148.
- Kraichnan, R.H. 1962. Mixing-length analysis of turbulent thermal convection at arbitrary Prandtl number, *Phys. Fluids* **5**, 1374–1389.
- Kraichnan, R.H. 1964a. Direct-interaction approximation for shear and thermally driven turbulence, *Phys. Fluids* **7**, 1048–1062.
- Kraichnan, R.H. 1964b. Kolmogorov’s hypotheses and Eulerian turbulence theory, *Phys. Fluids* **7**, 1723–1734.
- Kraichnan, R.H. 1965a. Lagrangian-history closure approximation for turbulence, *Phys. Fluids* **8**, 575–598.
- Kraichnan, R.H. 1965b. Inertial-range spectrum of hydrodynamic turbulence, *Phys. Fluids* **8**, 1385–1387.
- Kraichnan, R.H. 1966a. Isotropic turbulence and inertial-range structure, *Phys., Fluids* **9**, 1728–1752.
- Kraichnan, R.H. 1966b. Dispersion of particle pairs in homogeneous turbulence, *Phys. Fluids* **9**, 1937–1943.
- Kraichnan, R.H. 1967a. Condensate turbulence in a weakly coupled boson gas, *Phys. Rev. Lett.* **18**, 202–206.
- Kraichnan, R.H. 1967b. Inertial ranges in two-dimensional turbulence, *Phys. Fluids* **10**, 1417–1423.
- Kraichnan, R.H. 1967c. Intermittency in the very small scales of turbulence, *Phys., Fluids* **10**, 2081–2082.
- Kraichnan, R.H., 1968a. Lagrangian-history statistical theory for Burgers equation *Phys. Fluids* **11**, 266–277.
- Kraichnan, R.H. 1968b. Small-scale structure of a scalar field convected by turbulence, *Phys. Fluids* **11**, 945–953.
- Kraichnan, R.H. 1968c. Convergents to infinite series in turbulence theory. *Phys. Rev.* **174**, 240–246.
- Kraichnan, R.H. 1970a. Turbulent diffusion: evaluation of primitive and renormalized perturbation series by Padé approximants and by expansion of Stieltjes transforms into contributions from continuous orthogonal functions, in: *The Padé Approximant in Theoretical Physics*, Eds. Baker, G.A. & Gammel, J.L., Academic Press, New York, pp. 129–170.
- Kraichnan, R.H. 1970b. Instability in fully-developed turbulence. *Phys. Fluids* **13**, 569–575.
- Kraichnan, R.H. 1970c. Convergents to turbulence functions, *J. Fluid Mech.* **41**, 189–218.
- Kraichnan, R.H. 1971a. An almost-Markovian Galilean-invariant turbulence model, *J. Fluid Mech.* **47**, 513–524.
- Kraichnan, R.H. 1971b. Inertial-range transfer in two- and three-dimensional turbulence, *J. Fluid Mech.* **47**, 525–535.
- Kraichnan, R.H. 1972. Some modern developments in the statistical theory of turbulence, in *Statistical Mechanics; New Concepts, New Problems, New Applications*, eds. S. A. Rice, K. F. Freed and J. C. Light, University of Chicago Press, Chicago, pp. 201–228.
- Kraichnan, R.H. 1974a. On Kolmogorov’s inertial-range theories, *J. Fluid Mech.* **62**, 305–330.
- Kraichnan, R.H. 1974b. Passive-scalar convection by a quasi-uniform random straining field, *J. Fluid Mech.* **64**, 737–762.
- Kraichnan, R.H. 1975a. Remarks on turbulence theory, *Adv. Math.* **16**, 305–331.
- Kraichnan, R.H. 1975b. Statistical dynamics of two-dimensional flow, *J. Fluid Mech.* **67**, 155–175.
- Kraichnan, R.H. 1976a. Diffusion of weak magnetic fields by isotropic turbulence. *J. Fluid Mech.* **75**, 657–676.
- Kraichnan, R.H. 1976b. Eddy viscosity in two and three dimensions, *J. Atmos. Sci.* **33**, 1521–1536.
- Kraichnan, R.H. 1976c. Diffusion of passive-scalar and magnetic fields by helical turbulence. *J. Fluid Mech.* **77**, 753–768.
- Kraichnan, R.H. 1979a. Variational method in turbulence theory. *Phys. Rev. Lett.* **42**, 1263–1266.
- Kraichnan, R.H. 1979b. Consistency of the alpha-effect turbulent dynamo. *Phys. Rev. Lett.* **42**, 1677–1680.
- Kraichnan, R.H. 1985. Decimated amplitude equations in turbulence dynamics, in: *Theoretical Approaches to Turbulence*, Eds. Dwoyer, D.L., Hussaini, M.Y. and Voigt. R.G., Springer-Verlag, New York, pp. 91–135.
- Kraichnan, R.H. 1988. Reduced description of hydrodynamic turbulence. *J. Stat. Phys.* **51**, 949–963.
- Kraichnan, R.H. 1991. Turbulent cascade and intermittency growth, *Proc. Roy. Soc. Lond. A* **434**, 65–78.
- Kraichnan, R.H. 1994. Anomalous scaling of a randomly advected passive scalar, *Phys. Rev. Lett.* **72**, 1016–1019.
- Kraichnan, R.H. 1999. Note on forced Burgers turbulence. *Phys. Fluids* **11**, 3738–3742.

- Kraichnan, R.H. & Chen, S. 1989. Is there a statistical mechanics of turbulence? *Physica D* **37**, 160–172.
- Kraichnan, R.H., Chen, H. & Chen, S. 1989. Probability distribution of a stochastically advected scalar field. *Phys. Rev. Lett.* **63**, 2657–2657.
- Kraichnan, R.H. & Gotoh, T. 1993. Statistics of decaying Burgers turbulence. *Phys. Fluids A* **5**, 445–457.
- Kraichnan, R.H. & Leith, C.E. 1972. Predictability of turbulent flows. *J. Atmos. Sci.* **29**, 1041–1058.
- Kraichnan, R.H. & Montgomery, D. 1980. Two-dimensional turbulence, *Rep. Prog. Phys.* **43**, 547–618.
- Kraichnan, R.H. & Nagarajan, S. 1967. Growth of turbulent magnetic fields. *Phys. Fluids* **10**, 859–870.
- Kraichnan, R.H. & Orszag, S.A. 1967. Model equations for strong turbulence in a Vlasov plasma. *Phys. Fluids* **10**, 1720–1736.
- Landau, L.D. & Lifshitz, E.M. 1987. *Fluid Mechanics*, 2nd ed., Pergamon Press, Oxford.
- Lee, T.D. 1951. Difference between turbulence in a two-dimensional fluid and in a three-dimensional fluid, *J. Appl. Phys.* **22**, 524–524.
- Lee, T.D. 1952. On some statistical properties of hydrodynamic and hydromagnetic fields, *Quarterly Appl. Math.* **10**, 69–72.
- Leith, C.E. 1967. Diffusion approximation to inertial energy transfer in isotropic turbulence, *Phys. Fluids* **10**, 1409–1416.
- Lesieur, M. 2008. *Turbulence in Fluids*, 4th edition. Springer, Heidelberg.
- Leslie, D.C. 1973. *Developments in the Theory of Turbulence*, Oxford University Press, Clarendon.
- Lesieur, M., Frisch, U. & Brissaud, A. 1971. Théorie de Kraichnan de la turbulence. Application à l'étude d'une turbulence possédant de l'hélicité, *Ann. Géophys. (Paris)* **27**, 151–165.
- Lewis, R.M. & Kraichnan, R.H. 1962. A space-time functional formalism for turbulence, *Commun. Pure Appl. Math.* **15**, 397–411.
- Lighthill, M.J. 1952. On sound generated aerodynamically. I. general theory, *Proc. R. Soc. Lond. A* **211**, 564–587.
- Lighthill, M.J. 1953. On the energy scattered from the interaction of turbulence with sound or shock waves, *Proc. Camb. Phil. Soc.* **49**, 531–551.
- Lund, F. & Rojas, C. 1989. Ultrasound as a probe of turbulence, *Physica D* **37**, 508–514.
- Makeenko, Y.M. & Migdal, A.A. 1979. Exact equation for the loop average in multicolor QCD, *Phys. Lett. B* **88**, 135–137.
- Mandelbrot, B.B. 1968. On intermittent free turbulence. in: *Turbulence of Fluids and Plasmas. New York, April 16-18*. Brooklyn Polytechnic Inst., New York (abstract).
- Martin, P.C., Siggia, E.D. & Rose, H.A. 1973. Statistical dynamics of classical systems, *Phys. Rev. A* **8**, 423–437.
- Mandelbrot, B.B. 1974. Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier, *J. Fluid Mech.* **62**, 331–358.
- Migdal, A.A. 1983. Loop equations and 1/N expansion *Phys. Rep.* **102**, 199–290.
- Millionschikov, M. 1941. On the theory of homogeneous isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 615–618.
- Monin, A.S. & Yaglom, A.M. 1975. *Statistical Fluid Mechanics*, Vol. 2:, MIT Press, Cambridge, MA.
- Nazarenko, S.V. 2010. *Wave Turbulence*, Lect. Notes in Phys., Springer, Heidelberg, in press.
- Nazarenko, S.V., Newell, A. C. & Galtier, S. 2001. Non-local MHD turbulence, *Physica D* **152**, 646–652.
- Neumann, J. von. 1949. Recent theories of turbulence, in *Collected works (1949–1963)* **6**, 437–472, ed. A.H. Taub. Pergamon Press, New York (1963).
- Ng, C.S. & Bhattacharjee, A. 1996. Interaction of shear-Alfvén wave packets: implication for weak magnetohydrodynamic turbulence in astrophysical plasmas, *Astrophys. J.* **465**, 845–854.
- Niemela, J.J., Skrbek, L., Sreenivasan, K.R. & Donnelly, R.J. 2000. Turbulent convection at high Rayleigh numbers, *Nature* **404**, 837–841.
- Novikov E.A. & Stewart R. 1964. Intermittency of turbulence and spectrum of fluctuations in energy-dissipation, *Izv. Akad. Nauk. SSSR. Ser. Geofiz* **3**, 408–413.
- Obukhov, A.M. 1949. Structure of the temperature field in a turbulent flow, *Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz.* **13**, 58–69.
- Obukhov, A.M. 1962. Some specific features of atmospheric turbulence, *J. Fluid Mech.* **13**, 77–81.
- Ogura, Y. 1963. A consequence of the zero-fourth-cumulant approximation in the decay of isotropic turbulence, *J. Fluid Mech.* **16**, 33–40.
- Onsager, L. 1949. Statistical hydrodynamics, *Nuovo Cim. Suppl.* **6**, 279–287.
- Orszag, S.A. 1966. Dynamics of fluid turbulence, Princeton Plasma Physics Laboratory, report PPL-AF-13.

- Orszag, S.A. 1977. Statistical Theory of Turbulence, in *Fluid Dynamics, Les Houches 1973*, 237–374, eds. R. Balian & J.L. Peube. Gordon and Breach, New York
- Orszag, S.A. & Kruskal, M.D. 1966. Theory of turbulence, *Phys. Rev. Lett.* **16**, 441–444.
- Ott, S. & Mann, J. 2000. An experimental investigation of the relative diffusion of particle pairs in three-dimensional turbulent flow, *J. Fluid Mech.* **422**, 207–223.
- Parisi, G. & Frisch, U. 1985. On the singularity structure of fully developed turbulence, in: *Turbulence and Predictability in Geophysical Fluid Dynamics, Proceedings of the International School of Physics ‘E. Fermi’, 1983, Varenna, Italy*. Eds. M. Ghil, R. Benzi & G. Parisi, North Holland, Amsterdam, pp. 84–87.
- Pasquero C. & Falkovich, G. 2002. Stationary spectrum of vorticity cascade in two-dimensional turbulence, *Phys. Rev. E* **65**, 056305.
- Pierotti D. 1997. Intermittency in the large-N limit of a spherical shell model for turbulence *Europhys. Lett.* **37**, 323–328
- Pierotti, D., L’vov, V.S., Pomyalov, A. & Procaccia, I. 2000. Anomalous scaling in a model of hydrodynamic turbulence with a small parameter *Europhys. Lett.* **50**, 473–479.
- Prandtl, L. 1925. Bericht über Untersuchungen zur ausgebildeten Turbulenz. *Zs. angew. Math. Mech.* **5**, 136–139.
- Priestley, C.H.B. 1959. *Turbulent Transfer in the Lower Atmosphere*, University of Chicago Press, Chicago.
- Proudman, I. 1961. On Kraichnan’s theory of turbulence, in *Mécanique de la Turbulence. Colloques Internat. CNRS* **108**, 107–112, ed. A. Favre, CNRS, Paris.
- Reid, W. H. 1959. Tech. Rept. No.23, Division of Applied Mathematics, Brown University.
- Rhines, P.B. 1979. Geostrophic turbulence, *Ann. Rev. Fluid Mech.* **11**, 401–441.
- Rose, H.A. & Sulem, P.-L. 1978. Fully developed turbulence and statistical mechanics, *J. Phys. France*, 441–484.
- Rüdiger, G. 1974. The influence of a uniform magnetic field of arbitrary strength on turbulence, *Astron. Nachr.* **295**, 275–284.
- Salmon, R. 1980. Baroclinic instability and geostrophic turbulence, *Geophys. Astrophys. Fluid Dyn.* **15**, 167–211.
- Salmon, R. 1998. *Lectures on Geophysical Fluid Dynamics*, Oxford University Press US.
- Scott, R.B. & Wang, F. 2005. Direct evidence of an oceanic inverse kinetic energy cascade from satellite altimetry, *J. Phys. Ocean.* **35**, 1650–1666.
- Semikoz, D.V. & Tkachev, I.I. 1997. Condensation of bosons in the kinetic regime *Phys. Rev. D* **55**, 489–502.
- Shebalin, J.V., Matthaeus, W.H. & Montgomery, D. 1983. Anisotropy in MHD turbulence due to a mean magnetic field, *J. Plasma Phys.* **29**, 525–547.
- Shraiman, B.I. & Siggia, E.D. 1995. Anomalous scaling and small scale anisotropy of a passive scalar in turbulent flow, *C.R. Acad. Sci.* **321**, 279–284.
- Smith, L.M. & Yakhot, V. 1993. Bose condensation and small-scale structure generation in a random force driven 2D turbulence. *Phys. Rev. Lett.* **71**, 352–355.
- Spiegel, E.A. 1971. Convection in stars. *Annu. Rev. Astron. Astrophys.* **9**, 323–352.
- Spiegel, E.A. 2010. Chandrasekhar’s Lecture Notes on the Theory of Turbulence (1954). *Lect. Notes Phys.* Springer, Heidelberg, in press.
- Starr, V.P. 1968. *Physics of Negative Viscosity Phenomena*. McGraw-Hill, New York
- Tatsumi, T. 1957. The theory of decay processes of incompressible isotropic turbulence. *Proc. R. Soc. Lond. A* **239**, 16–45.
- Taylor, G.I. 1917. Observations and speculations on the nature of turbulence motion, Reports and Memoranda of the Advisory Committee for Aeronautics, no. 345; reproduced in *G.I. Taylor’s Scientific Papers*, ed. G.K. Batchelor, Cambridge University Press, 1960, vol. II, paper No. 7, pp. 69–78.
- Ting, L. & Miksis, M.J. 1990. On vortical flow and sound generation, *Siam. J. Appl. Math.* **50**, 521–536.
- Xiao, Z., Wan, M., Chen, S. & Eyink, G.L. 2008. Physical mechanism of the inverse energy cascade of two-dimensional turbulence: a numerical investigation. *J. Fluid Mech.* **619**, 1–44.
- Yaglom, A.M. 1949. Local structure of the temperature field in a turbulent flow, *Dokl. Akad. Nauk SSSR* **69**, 743–746.
- Yaglom, A.M. 1966. The influence of fluctuations in energy dissipation on the shape of turbulence characteristics in the inertial interval. *Dokl. Akad. Nauk. SSSR* **166**, 49–52.
- Zakharov, V.E. 1966. Some aspects of nonlinear theory of surface waves (in Russian), PhD Thesis, Budker Institute for Nuclear Physics for Nuclear Physics, Novosibirsk, USSR.
- Zakharov, V., L’vov, V. & Falkovich, G. 1992. *Kolmogorov Spectra of Turbulence*. Springer-Verlag, Berlin.